

Some characterizations of self-adjoint operators

FUAD KITTANEH

Let H be a separable, infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H . For $T \in B(H)$, the absolute part of T , denoted by $|T|$, is defined as usual as the positive square root of T^*T . Each $T \in B(H)$ can be uniquely expressed as $A + iB$, where A, B are self-adjoint operators called the real part and the imaginary part respectively, denoted by $\operatorname{Re}T$ and $\operatorname{Im}T$, respectively. Note that $\operatorname{Re}T = (T + T^*)/2$ and $\operatorname{Im}T = (T - T^*)/2i$.

The following two theorems are characterization of self-adjoint and positive operators and were obtained by FONG and ISTRATESCU [1] and FONG and TSUI [2], respectively.

Theorem A. *An operator $T \in B(H)$ is self-adjoint if and only if $|T|^2 \leq (\operatorname{Re}T)^2$.*

Theorem B. *An operator $T \in B(H)$ is positive if and only if $|T| \leq \operatorname{Re}T$.*

The purpose of this note is to generalize Theorem A as well as to present a new proof of Theorem B which may lead to further development in this direction. At the end of this paper we will give some characterization modulo C_p (the Schatten p -class) of self-adjoint operators.

Recall that $T \in B(H)$ is said to be hyponormal if $TT^* \leq T^*T$ and in this case the spectral radius $r(T) = \|T\|$ (see [6]).

Theorem 1. *Let $T \in B(H)$ be hyponormal. If for some $S \in B(H)$ and a complex number α , $|T|^2 + \alpha(TS - ST) \leq 0$, then $T = 0$.*

Proof. Since $r(T) = \|T\|$, there exists a sequence $\{x_n\}$ of unit vectors in H such that $(T - t)x_n \rightarrow 0$ where $|t| = \|T\|$. Now $(T^*Tx_n, x_n) + \alpha(TSx_n, x_n) - \alpha(STx_n, x_n) \leq 0$. Hence $\|Tx_n\|^2 + \alpha(Sx_n, (T - t)^*x_n) - \alpha((T - t)x_n, S^*x_n) \leq 0$. But since T is hyponormal and $\|(T - t)x_n\| \rightarrow 0$, it follows that $\|(T - t)^*x_n\| \rightarrow 0$. Letting $n \rightarrow \infty$, in the last inequality, we obtain $|t|^2 \leq 0$. Hence $T = 0$ as required.

Received January 17, 1983.

Corollary 1. *If $|T|^2 \leq (\operatorname{Re} T)^2$, then $T = T^*$.*

Proof. Let $T = A + iB$. Then $|T|^2 \leq (\operatorname{Re} T)^2$ is equivalent to $B^2 + i(AB - BA) \leq 0$. Now the corollary follows from Theorem 1.

The following proof of Theorem B was suggested to me by J. Stampfli.

Lemma 1. *Let $T \in \mathcal{B}(H)$ be such that $T = VP$ where V is a contraction, $P \geq 0$ and $2P \leq VP + PV^*$. Let $P = D + K$ where D is diagonal and positive and K is arbitrary. If $Dx = \lambda x$ with x a unit vector in H and $\lambda > 0$, then $\|(1 - V)^*x\| \leq (2/\lambda)\|Kx\|$.*

Proof. Observe first that $\|(1 - V)^*x\|^2 \leq 2 - ((V + V^*)x, x)$. Now

$$\begin{aligned} 2\lambda + 2(Kx, x) &= 2(Px, x) \leq (VPx, x) + (PV^*x, x) = \\ &= (VDx, x) + (DV^*x, x) + (VKx, x) + (KV^*x, x) = \\ &= \lambda((V + V^*)x, x) + (VKx, x) + (KV^*x, x). \end{aligned}$$

Therefore $\lambda[2 - ((V + V^*)x, x)] \leq ((V - 1)Kx, x) + (K(V^* - 1)x, x)$ and so

$$2 - ((V + V^*)x, x) \leq (2/\lambda) \|(V - 1)^*x\| \|Kx\|.$$

Combining this inequality with the first inequality, we obtain

$$\|(1 - V)^*x\| \leq (2/\lambda) \|Kx\|$$

as required.

An alternative proof of Theorem B. Let $T = VP$ be the polar decomposition of T . Let $P = \int t dE(t)$ where $E(t)$ is the spectral measure of P . Fix $\alpha > 0$ and let $H_\alpha = E([\alpha, \infty))H$. If $\varepsilon > 0$ is given, then by Weyl—Von Neumann Theorem [3], $P_\alpha = D + K$ where D is diagonal and K is Hilbert—Schmidt with $\|K\|_2 < \varepsilon$ ($\|\cdot\|_2$ denotes the Hilbert—Schmidt norm). If $De_n = \lambda_n e_n$ where $\{e_n\}$ is a basis for H_α , then for any unit vector $y \in H_\alpha$, $y = \sum_{n=1}^{\infty} a_n e_n$ for some a_n with $\sum_{n=1}^{\infty} |a_n|^2 = 1$. Applying the lemma, we obtain

$$\begin{aligned} \|(1 - V)^*y\| &\leq \sum_{n=1}^{\infty} \|(1 - V)^*a_n e_n\| \leq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \|(1 - V)^*e_n\|^2 \right)^{1/2} \leq \\ &\leq (2/\alpha) \left(\sum_{n=1}^{\infty} \|Ke_n\|^2 \right)^{1/2} < (2/\alpha)\varepsilon. \end{aligned}$$

Since ε is arbitrary, $V = 1$ on H_α . Since $\alpha > 0$ is arbitrary we have $V = 1$ on $(\ker P)^\perp = \overline{R(P)}$. Therefore $T = VP = P \geq 0$ as required.

We remark that the above proof works for the following generalization of Theorem B.

Theorem 2. *If $P \geq 0$, V is a contraction and $2P \leq VP + PV^*$, then $P = VP$ and $V|_{(\ker P)^\perp} = 1$.*

In what follows we shall prove that if $|T|^2 - (\operatorname{Re} T)^2 \in C_p$ ($p \geq 1$), then $T - T^* \in C_{2p}$. Recall that a compact operator C is in C_p if and only if $\|C\|_p^p = \sum_{i=1}^{\infty} s_i(C)^p < \infty$ where $s_1(C), s_2(C), \dots$ denotes the sequence of eigenvalues of $|C|$ in decreasing order and repeated according to multiplicity. It is known (see [7]) that for $p \geq 1$, $\|C\|_p^p \cong \sum_{n=1}^{\infty} |(Ce_n, f_n)|^p$ for any orthonormal sets $\{e_n\}$ and $\{f_n\}$ in H . We refer to [5] or [7] for further properties of the Schatten p -classes.

Lemma 2. *Let $T \in B(H)$ be hyponormal. If for some $S \in B(H)$ and a complex number α , $|T|^2 + \alpha(TS - ST)$ is compact, then T is compact.*

Proof. Let $K(H)$ denote the closed ideal of compact operators in $B(H)$, and let $\pi: B(H) \rightarrow B(H)/K(H)$ be the quotient map of $B(H)$ onto the Calkin algebra $B(H)/K(H)$. Therefore $|\pi(T)|^2 + \alpha(\pi(T)\pi(S) - \pi(S)\pi(T)) = 0$ and so by Theorem 1 we have $\pi(T) = 0$, in other words, T is compact. (Recall that the Calkin algebra is a B^* -algebra and so it is representable as an operator algebra.)

Theorem 3. *Let $T \in B(H)$ be hyponormal. If for some $S \in B(H)$ and a complex number α , $|T|^2 + \alpha(TS - ST) \in C_p$ ($p \geq 1$), then $T \in C_{2p}$.*

Proof. Since $C_p \subset K(H)$, we have by Lemma 2 that $T \in K(H)$. But it is known [6] that a compact hyponormal operator is diagonal, therefore $Te_n = \lambda_n e_n$ for some basis $\{e_n\}$ of H . Thus

$$\begin{aligned} \infty &> \| |T|^2 + \alpha(TS - ST) \|_p^p \cong \sum_{n=1}^{\infty} \left| (|T|^2 + \alpha(TS - ST)e_n, e_n) \right|^p = \\ &= \sum_{n=1}^{\infty} \left| \|Te_n\|^2 + \alpha(Se_n, T^*e_n) - \alpha(Te_n, S^*e_n) \right|^p = \\ &= \sum_{n=1}^{\infty} \left| |\lambda_n|^2 + \alpha(Se_n, e_n) - \alpha(e_n, S^*e_n) \right|^p = \sum_{n=1}^{\infty} |\lambda_n|^{2p} \end{aligned}$$

and so $T \in C_{2p}$ as required.

Corollary. *If $|T|^2 - (\operatorname{Re} T)^2 \in C_p$ ($p \geq 1$), then $T - T^* \in C_{2p}$. Hence T has a non-trivial invariant subspace.*

Proof. Observe that $|T|^2 - (\operatorname{Re} T)^2 = B^2 + i(AB - BA) \in C_p$ and apply Theorem 3 to get $B \in C_{2p}$. The last assertion follows from Corollary 6.15 in [4] (which says that if $T - T^* \in C_p$ for some $p \geq 1$, then T has a non-trivial invariant subspace).

Acknowledgment. I would like to thank Professor J. G. Stampfli for his kind assistance.

References

- [1] C. K. FONG and V. I. ISTRATESCU, Some characterization of Hermitian operators and related classes of operators. I, *Proc. Amer. Math. Soc.*, **76** (1979), 107—112.
- [2] C. K. FONG and S. K. TSUI, A note on positive operators, *J. Operator Theory*, **5** (1981), 73—76.
- [3] J. VON NEUMANN, *Charakterisierung des spektrums eines integraloperators*, Actualities Sci. Indust., No. 229, Hermann (Paris, 1935).
- [4] H. RADJAVI and P. ROSENTHAL, *Invariant subspaces*, Springer-Verlag (Berlin, 1973).
- [5] J. R. RINGROSE, *Compact non-self-adjoint operators*, Van Nostrand Reinhold Co. (New York, 1971).
- [6] J. G. STAMPFLI, Hyponormal operators, *Pacific J. Math.*, **12** (1962), 1453—1458.
- [7] B. SIMON, *Trace ideals and their applications*, University Press (Cambridge, 1979).

DEPARTMENT OF MATHEMATICS
UNITED ARAB EMIRATES UNIVERSITY
P. O. BOX: 15551
AL-AIN, U.A.E.