

The Sz.-Nagy “théorème principal” extended. Application to subnormality

F. H. SZAFRANIEC

*A Monsieur le Professeur Béla Szőkefalvi-Nagy
à l'occasion de son quatre-vingtième anniversaire*

The purpose of this paper¹⁾ is to extend the celebrated Sz.-Nagy general dilation theorem of [24] from bounded operators to unbounded ones. It turns out to be convenient to replace, like in [18], bounded operator valued functions on $*$ -semigroups by forms. So as to get complete description of the shift operator determined by such a form the reproducing kernel Hilbert space construction is used in the way proposed in [8].

The Hilbert spaces to be considered here are complex ones and $\langle \bullet, - \rangle$ always stands for inner product regardless the Hilbert space involved is; we have to trust this leads to no confusion. $B(\mathcal{H})$ denotes the algebra of all bounded linear operators in $\mathcal{H}$ and I stands for the identity operator. $\mathcal{D}(A)$ denotes the domain of unbounded densely defined operator A , A^* denotes its adjoint, A^- its closure. Semigroups considered here are, in general, in multiplicative notation; if a semigroup $\mathfrak{G}$ has involution, that is a mapping $s \rightarrow s^*$ satisfying $(st)^* = t^*s^*$, $s, t \in \mathfrak{G}$ and $e^* = e$ provided $\mathfrak{G}$ has a unit element e , then it is called $*$ -semigroup.

1. Starting from 1955

In that year the celebrated Béla Szőkefalvi-Nagy Appendix “Prolongment des transformations de l'espace de Hilbert qui sortent de cet espace” appeared; it centers around the famous “théorème principal”. This general dilation theorem

Received September 14, 1992.

¹⁾ The research resulting in this paper has been supported by the grant of KBN (Komitet Badań Naukowych, Warsaw).

was used there to deduce a number of diverse results in Operator Theory. Let us state here, in different notation, the substantial part of this theorem. A map $\varphi: \mathfrak{G} \rightarrow B(\mathcal{H})$, $\mathfrak{G}$ being a $*$ -semigroup is said to be *positive definite* for any finite sequences $s_0, \dots, s_N \in \mathfrak{G}$ and $x_0, \dots, x_N \in \mathcal{H}$

$$\sum_{i,j=0}^N \langle \varphi(s_j^* s_i) x_i, x_j \rangle \geq 0.$$

The Sz.-Nagy “théorème principal” [24]. *Let $\mathfrak{G}$ be a $*$ -semigroup with a unit. Then a necessary and sufficient condition for $\varphi: \mathfrak{G} \rightarrow B(\mathcal{H})$ to be represented as*

$$\varphi(s) = V^* \Phi(s) V, \quad s \in \mathfrak{G}$$

where V is a bounded linear map of $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ and Φ is a unital involution preserving semigroup homomorphism of $\mathfrak{G}$ into $B(\mathcal{K})$, is that φ be positive definite and satisfy the boundedness condition

$$\sum_{i,j=0}^N \langle \varphi(s_j^* u^* u s_i) x_i, x_j \rangle \leq c(u) \sum_{i,j=0}^N \langle \varphi(s_j^* s_i) x_i, x_j \rangle$$

with $c(u)$ independent of s_i and x_i . If $\varphi(e) = I$, V becomes an isometry.

It turns out that the Sz.-Nagy general dilation theorem is equivalent to another famous result of those times, namely to the Stinespring decomposition theorem, cf. [20]. Reporting briefly²⁾ on what has happened else around this theorem we want to mention results simplifying the boundedness condition, cf. [14], [16], [17], [9] as well as [25] and a version of this theorem in the nonunital case [15]. It may also be interesting to record the result of [26] showing that in some circumstances the boundedness condition replaces positive definiteness.

2. Recalling the RKHS construction

We wish to collect some facts from the theory of reproducing kernel Hilbert spaces, cf. [1]. Let A be a set; a function $K: A \times A \rightarrow \mathbb{C}$ is referred to as a *kernel*.

²⁾ We do not pretend to mention here all the works related, in particular we do not quote those which go out of the Hilbert space scope unless this is necessary.

The kernel K is said to be positive definite if for any finite sequences $a_0, \dots, a_N \in A$ and $\xi_0, \dots, \xi_N \in \mathbb{C}$

$$\sum_{i,j=0}^N K(a_i, a_j) \xi_i \bar{\xi}_j \geq 0.$$

Set $K_a = K(\bullet, a)$. Then the reproducing kernel Hilbert space³⁾ $\mathcal{H}_K$ of the kernel K is composed of functions $f: A \rightarrow \mathbb{C}$, the linear span of the functions K_a , $a \in A$ is dense in $\mathcal{H}_K$ and the inner product of $\mathcal{H}_K$ extends from

$$\langle K_a, K_b \rangle = K(a, b), \quad a, b \in A.$$

Besides the Schwarz inequality there are two useful tools: the reproducing kernel property and, its consequence, the RKHS test. The *reproducing kernel property* says that for any $f \in \mathcal{H}_K$

$$f(a) = \langle K_a, f \rangle, \quad a \in A.$$

The *reproducing kernel Hilbert space test*, cf. [19], asserts that f belongs to $\mathcal{H}_K$ if and only if there is $c = c_f$ such that for any finite sequences $a_0, \dots, a_N \in A$ and $\xi_0, \dots, \xi_N \in \mathbb{C}$

$$|\sum_{i=0}^N f(a_i) \xi_i|^2 \leq c \sum_{i,j=0}^N K(a_i, a_j) \xi_i \bar{\xi}_j.$$

3. Extending the dilation theorem

Suppose $\mathcal{E}$ is a complex linear space and $\mathfrak{G}$ is a semigroup with unit and consider a mapping $K: \mathfrak{G} \times \mathfrak{G} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{C}$. We think of K as a kernel on $\mathfrak{G} \times \mathcal{E}$ (arranging $\mathfrak{G} \times \mathfrak{G} \times \mathcal{E} \times \mathcal{E}$ in the appropriate order if one needs this). Suppose that for any $s, t \in \mathfrak{G}$ $K(s, t, \bullet, -)$ is a hermitian bilinear (=sesquilinear) form⁴⁾ on $\mathcal{E}$; this justifies the name *form kernel over* $(\mathfrak{G}, \mathcal{E})$ we intend to use here. Thus positive definiteness of K can be written explicitly as

$$\sum_{i,j=0}^N K(s_i, s_j, x_i, x_j) \geq 0$$

³⁾ We use the abbreviation RKHS.

⁴⁾ We follow here the idea, implemented in [8], of using a scalar RKHS in the context of (unbounded) operator valued kernels.

for any finite sequences $s_0, \dots, s_N \in \mathfrak{G}$ and $x_0, \dots, x_N \in \mathcal{E}$. Adopt the notation of the preceding section, in particular for $K_{s,x}$ and $\mathcal{H}_K$. Thus members of $\mathcal{H}_K$ are functions $f: \mathfrak{G} \times \mathcal{E} \rightarrow \mathbb{C}$ and because they are pointwise limits of functions $\sum_{i=0}^N \xi_i K_{s_i, x_i}$, the mappings $x \rightarrow \overline{f(s, x)}$, $s \in \mathfrak{G}$ are always linear.

Set

$$V: \mathcal{E} \ni x \rightarrow K_{e,x} \in \mathcal{H}_K.$$

If K is *nondegenerate*, that is if $K(e, e, x, y) = 0$ for any y implies $x = 0$, or, equivalently $K(e, e, x, x) = 0$ implies $x = 0$, then the linear mapping V is injective. If $\mathcal{E}$ is a normed space and K is *isometric*, that is $K(e, e, x, x) = \|x\|$, $x \in \mathcal{E}$, then V is an isometry if $\mathcal{E}$ into $\mathcal{H}_K$. If $\mathcal{E}$ is a dense subspace of a Hilbert space $\mathcal{H}$, say, and K is isometric, then V extends to $\mathcal{H}$ and the resulting operator imbeds isometrically $\mathcal{H}$ into $\mathcal{H}_K$. Notice that in this case the form φ must necessarily come from an (unbounded) operator valued kernel on $\mathfrak{G}$, more precisely, there are linear mappings $k(s, t): \mathcal{E} \rightarrow \mathcal{H}$, $s, t \in \mathfrak{G}$ such that $K(s, t, x, y) = \langle k(s, t)x, y \rangle$, $x, y \in \mathcal{E}$. The operators $k(s, t)$ are, in general unbounded; they are bounded if and only if, for any $f \in \mathcal{H}_K$ and $s \in \mathfrak{G}$, the functional $x \rightarrow \overline{f(s, x)}$, which as we already know is always linear, is bounded⁵⁾.

Denote by $\mathcal{D}$ the linear span of $K_{s,x}$, $s \in \mathfrak{G}$, $x \in \mathcal{E}$ and, for $u \in \mathfrak{G}$, by $\mathcal{D}(u)$ the linear space of $f \in \mathcal{H}_K$ such that $f_u \in \mathcal{H}_K$ too, where

$$f_u(s, x) = f(us, x), \quad s \in \mathfrak{G}, \quad x \in \mathcal{E}.$$

There are two natural candidates for the shift operator: Φ_u and $\Phi(u)$. Define them as follows

$$\begin{aligned} \mathcal{D}(\Phi_u) &= \mathcal{D}, & \Phi_u K_{s,x} &= K_{us,x}, & s \in \mathfrak{G}, \quad x \in \mathcal{E}, \\ \mathcal{D}(\Phi(u)) &= \mathcal{D}(u), & \Phi(u)f &= f_u, & f \in \mathcal{D}(u). \end{aligned}$$

The first operator is well defined if and only if

$$\sum_{i,j=0}^N K(s_i, s_j, x_i, x_j) = 0 \quad \text{implies} \quad \sum_{i,j=0}^N K(us_i, us_j, x_i, x_j) = 0,$$

while the second is always.

Set $\tilde{\Phi}_u = \Phi(u)|_{\mathcal{D}}$. Keeping up all the above notations we are ready to present our general dilation result.

⁵⁾ This can be deduced straightforwardly or get from [6]. Because this fact is not involved in any argument here, we do not discuss it in detail.

Theorem 1. *Let $\mathfrak{G}$ be a semigroup with unit and $\mathcal{E}$ a dense linear subspace of a Hilbert space $\mathcal{H}$. If K is an isometric positive definite form kernel over $(\mathfrak{G}, \mathcal{E})$ and the operators Φ_u , $u \in \mathfrak{G}$, are well defined, then⁶⁾*

$$K(s, t, x, y) = \langle \Phi_s Vx, \Phi_t Vy \rangle, \quad s, t \in \mathfrak{G}, \quad x, y \in \mathcal{E}$$

with $V: \mathcal{H} \rightarrow \mathcal{H}_K$ being isometry. Moreover, the mapping $s \rightarrow \Phi_s$ is multiplicative on $\mathcal{D}$, that is

$$\Phi_u \Phi_v f = \Phi_{uv} f, \quad u, v \in \mathfrak{G}, \quad f \in \mathcal{D}$$

while the mapping $s \rightarrow \Phi(s)$ is antimultiplicative, that is

$$\Phi(u)\Phi(v)f = \Phi(vu)f, \quad u, v \in \mathfrak{G}, \quad f \in \mathcal{D}(\Phi(u)\Phi(v)).$$

In addition to this

$$\Phi_u^* = \Phi(u), \quad u \in \mathfrak{G}$$

as well as

$$(1) \quad \mathcal{D}(\tilde{\Phi}_u^*) = \mathcal{D}(\Phi_u^-) + \mathcal{M}_1(u) \quad \text{and} \quad \mathcal{D}(u) = \mathcal{D}(\tilde{\Phi}_u^-) + \mathcal{M}_2(u)$$

where $\mathcal{M}_1(u) = \ker(I + \Phi(u)\tilde{\Phi}_u^*)$, $\mathcal{M}_2(u) = \ker(I + \tilde{\Phi}_u^*\Phi(u))$ and the sum being direct.

In the above $\Phi_u^* = (\Phi_u)^*$ and $\Phi_u^- = (\Phi_u)^-$.

Proof. The only things which need some proof are: $\Phi_s^* = \Phi(s)$ and the direct sum decompositions (1); the rest can be deduced straightforwardly from the construction.

First we want to prove that $\Phi_s^* = \Phi(s)$. Let $f \in \mathcal{H}_K$. Then it belongs to $\mathcal{D}(\Phi_u^*)$ if and if

$$|\langle \Phi_u \sum_{i=0}^N \xi_i K_{s_i, x_i}, f \rangle| \leq c_f \left\| \sum_{i=0}^N \xi_i K_{s_i, x_i} \right\|.$$

If f_u belongs to $\mathcal{H}_K$, then the reproducing kernel property used twice gives us

$$\langle \Phi_u \sum_{i=0}^N \xi_i K_{s_i, x_i}, f \rangle = \langle \sum_{i=0}^N \xi_i K_{us_i, x_i}, f \rangle = \sum_{i=0}^N \xi_i f(us_i, x_i) = \langle \sum_{i=0}^N \xi_i K_{s_i, x_i}, f_u \rangle$$

⁶⁾ Our aim is to present a construction of the objects appearing in the theorem. This is why we state it as a “one way” implication; the converse is easy both to formulate and to prove.

and this implies the aforesaid inequality with $c_f = \|f_u\|$. Conversely, because, as we have just showed,

$$\langle \Phi_u \sum_{i=0}^N \xi_i K_{s_i x_i}, f \rangle = \sum_{i=0}^N \xi_i f(us_i, x_i),$$

this inequality forces the RKHS test to tell us that $f_u \in \mathcal{H}_K$. This gives us $\Phi_s^* = \Phi(s)$.

Because $\Phi_u \subset \tilde{\Phi}_u^*$ and $\tilde{\Phi}_u \subset \Phi_u^*$, we apply Theorem 1 of [2] to the closure of the operators Φ_u and $\tilde{\Phi}_u$ to prove the direct sum decompositions (1). Because $\Phi_u^{-*} = \Phi_u^* = \Phi(u)$ and $\tilde{\Phi}_u^{-*} = \tilde{\Phi}_u^*$, the linear subspaces $\mathcal{M}_1(u)$ and $\mathcal{M}_2(u)$ are of the required form. ■

It is interesting to isolate the following fact as

Corollary⁷⁾ 1. $\mathcal{D}(\tilde{\Phi}_u^-) = \mathcal{D}(\Phi_u^-)$ if and only if $\ker(I + \Phi(u)\tilde{\Phi}_u^*) = \{0\}$. $\mathcal{D}(u) = \mathcal{D}(\tilde{\Phi}_u^-)$ if and only if $\ker(I + \tilde{\Phi}_u^*\Phi(u)) = \{0\}$.

Suppose the kernel K has the property that for any $u \in \mathfrak{G}$ there is a linear operator $A(u): \mathcal{E} \rightarrow \mathcal{E}$ such that

$$(2) \quad K(us, t, x, y) = K(s, t, x, A(u)y), \quad s, t \in \mathfrak{G}, \quad x, y \in \mathcal{E}.$$

(Notice that for a nondegenerate kernel the operators $A(u)$ are uniquely determined.) Because of hermitian symmetry of K this is the same as

$$K(s, ut, x, y) = K(s, t, A(u)x, y), \quad s, t \in \mathfrak{G}, \quad x, y \in \mathcal{E}.$$

Thus (2) implies

$$K(us, vt, x, y) = K(s, t, A(v)x, A(u)y), \quad s, t \in \mathfrak{G}, \quad x, y \in \mathcal{E}$$

and

$$(K_{s,x})_u = K_{s,A(u)x}.$$

Consequently, for $f \in \mathcal{D}(\tilde{\Phi}_u^*)$,

$$\begin{aligned} \langle \tilde{\Phi}_u^* f, K_{s,x} \rangle &= \langle f, \tilde{\Phi}_u K_{s,x} \rangle = \langle f, (K_{s,x})_u \rangle = \\ &= \langle f, K_{s,A(u)x} \rangle = f(s, A(u)x) = \langle f^u, K_{s,x} \rangle \end{aligned}$$

where

$$(3) \quad f^u(s, x) = f(s, A(u)x).$$

Thus we have arrived at the following

⁷⁾ Cf. [3], Satz 1.

Theorem 2. *If the form kernel K satisfies assumptions of Theorem 1 and has the property (2), then, in addition to the conclusion of Theorem 1,*

$$\tilde{\Phi}_u^* f = f^u, \quad f \in \mathcal{D}(\tilde{\Phi}_u^* f)$$

where f^u is defined by (3) and

$$\tilde{\Phi}_u V = VA(u).$$

The second equality of the conclusion can be deduced directly.

Consider the following condition

$$(4) \quad f(us, A(u)x) = -f(s, x), \quad s \in \mathfrak{G}, \quad x \in \mathcal{E}.$$

Then Corollary 1 implies now immediately

Corollary 2. $\mathcal{D}(\tilde{\Phi}_u^*) = \mathcal{D}(\Phi_u^-)$ if and only if (4) holds true for $f \in \mathcal{D}(\Phi(u)\tilde{\Phi}_u^*)$.
 $\mathcal{D}(u) = \mathcal{D}(\tilde{\Phi}_u^-)$ if and only if (4) holds true for $f \in \mathcal{D}(\tilde{\Phi}_u^* \Phi(u))$.

Suppose now that $\mathfrak{G}$ is a $*$ -semigroup with unit, call a mapping $\omega: \mathfrak{G} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{C}$ a *form function over* $(\mathfrak{G}, \mathcal{E})$ if, for any $s \in \mathfrak{G}$, $\omega(s, \bullet, -)$ is a hermitian bilinear (=sesquilinear) form on $\mathcal{E}$. A form function ω is said to be *positive definite* if for any finite sequences $s_0, \dots, s_N \in \mathfrak{G}$ and $x_0, \dots, x_N \in \mathcal{E}$

$$\sum_{i,j=0}^N \omega(s_j^* s_i, x_i, x_j) \geq 0.$$

This means that the form kernel⁸⁾ K on $A = \mathfrak{G} \times \mathcal{E}$ defined as

$$K(s, x, t, y) = \omega(t^* s, x, y)$$

is positive definite; denote the corresponding functions $K_{s,x}$ by $\omega_{s,x}$ and the resulting RKHS by $\mathcal{H}_\omega$. Notice that the operator⁹⁾ Φ_u is well defined, because from the Schwarz inequality we get

$$\left\| \sum_{i=0}^N \xi_i \omega_{us_i, x_i} \right\|^2 \leq \left\| \sum_{i=0}^N \xi_i \omega_{s_i, x_i} \right\| \left\| \sum_{i=0}^N \xi_i \omega_{u^* us_i, x_i} \right\|.$$

Now we can say more about Φ_u , $\tilde{\Phi}_u$ and $\Phi(u)$: the interplay of them becomes more symmetric. Theorem 1 has its version in the following

⁸⁾ What is in fact in Sz.-Nagy's proof [24], though not mentioned by name, is a RKHS of a (bounded) operator valued kernel, like it was exploited in Pedrick's unpublished Ph.D. thesis [6]. Later Masani used the same idea in [4] presenting a theorem reflecting his way of thinking of stochastic processes.

⁹⁾ It has, in fact, a prototype in [24].

Theorem 3. *Let $\mathfrak{G}$ be a $*$ -semigroup with unit and $\mathcal{E}$ a dense linear subspace of a Hilbert space $\mathcal{H}$. If ω is an isometric positive definite form over $(\mathfrak{G}, \mathcal{E})$, then¹⁰⁾*

$$\omega(s, x, y) = \langle \Phi_s Vx, Vy \rangle, \quad s \in \mathfrak{G} \quad x, y \in \mathcal{E}$$

with $V: \mathcal{H} \rightarrow \mathcal{H}_\omega$ being isometry. Moreover, the mapping $s \rightarrow \Phi_s$ is multiplicative on $\mathcal{D}$, that is

$$\Phi_u \Phi_v f = \Phi_{u,v} f, \quad u, v \in \mathfrak{G}, \quad f \in \mathcal{D}$$

while the mapping $s \rightarrow \Phi(s)$ is antimultiplicative, that is

$$\Phi(u)\Phi(v)f = \Phi(vu)f, \quad u, v \in \mathfrak{G}, \quad f \in \mathcal{D}(\Phi(u)\Phi(v)).$$

In addition to this

$$\Phi_u^* = \Phi(u), \quad \Phi_u = \Phi(u^*)|_{\mathcal{D}} \quad \text{and} \quad \tilde{\Phi}_u = \Phi_{u^*}$$

as well as

$$\mathcal{D}(\Phi(u^*)) = \mathcal{D}(\Phi_u^-) + \mathcal{M}(u)$$

where $\mathcal{M}(u) = \ker(I + \Phi(u)\Phi(u^))$, the sum being direct.*

Notice also that

$$\mathcal{D}(\Phi(u)\Phi(v)) \subset \mathcal{D}(\Phi(vu))$$

and, consequently

$$\ker(I + \Phi(u)\Phi(u^*)) \subset \ker(I + \Phi(u^*u)).$$

What is new in this Theorem can be obtained by direct calculation if one realizes the fact that

$$K(us, t, x, y) = K(s, u^*t, x, y).$$

Corollary 1 implies now

Corollary¹¹⁾ 3. $\Phi_u^- = \Phi(u^*)$ if and only if $\ker(I + \Phi(u)\Phi(u^*)) = \{0\}$.

Corollary 4. $\Phi_u^- = \Phi(u^*)$ if $\ker(I + \Phi(u^*u)) = \{0\}$.

¹⁰⁾ Our aim is to present a construction of the objects appearing in the theorem. This is why we state it as a “one way” implication; the converse is easy both to formulate and to prove.

¹¹⁾ Cf. [3], Satz 1 again.

4. Characterizing unbounded subnormal operators

An operator N is said to be *formally normal* if $\mathcal{D}(N) \subset \mathcal{D}(N^*)$ and $\|Nf\| = \|N^*f\|$ for $f \in \mathcal{D}(N)$; N is said to be *normal* if $\mathcal{D}(N) = \mathcal{D}(N^*)$. We get directly from definition of formal normality that

$$N^*Nx = NN^*x,$$

for $x \in (\mathcal{D}(N) \cap \mathcal{D}(N^*N)) \cap (\mathcal{D}(N^*) \cap \mathcal{D}(NN^*))$ while for a normal N the set of all x 's for which this equality holds is precisely $\mathcal{D}(N^2)$, cf. [23]; the latter set is always dense.

An operator S in $\mathcal{H}$ is said to be *subnormal* if there exists another Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $\mathcal{K}$ such that

$$\mathcal{D}(S) \subset \mathcal{D}(N) \cap \mathcal{H} \quad \text{and} \quad Sx = Nx, \quad x \in \mathcal{D}(S).$$

Set $\mathcal{D}^\infty(S) = \bigcap_{n=0}^\infty \mathcal{D}(S^n)$. From the aforesaid commutativity property one may deduce immediately that both formally normal and subnormal operators have necessarily to satisfy the inequality

$$(5) \quad \sum_{m,n=0}^M \langle S^m x_n, S^n x_m \rangle \geq 0, \quad x_0, \dots, x_M \in \mathcal{D}^\infty(S), \quad M = 0, 1, \dots$$

In contrast to the bounded case, even if S has invariant domain, e.g. $S\mathcal{D}(S) \subset \mathcal{D}(S)$ (then $\mathcal{D}^\infty(S) = \mathcal{D}(S)$) (5) may not force subnormality¹²⁾. The only thing which can be deduced from (5) is in the following

Proposition [12]. *Suppose $S\mathcal{D}(S) \subset \mathcal{D}(S)$. Then S satisfies (5) if and only if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a formally normal operator N in $\mathcal{K}$ such that*

$$\begin{aligned} \mathcal{D}(N) &= \lim \{N^{*n}x; x \in \mathcal{D}(S), n \geq 0\}, \quad N\mathcal{D}(N) \subset \mathcal{D}(N), \quad N\mathcal{D}(N^*) \subset \mathcal{D}(N), \\ \mathcal{D}(S) &\subset \mathcal{D}(N) \cap \mathcal{H} \quad \text{and} \quad Sx = Nx, \quad x \in \mathcal{D}(S). \end{aligned}$$

Following [13], a formally normal extension N of an operator S with invariant domain $\mathcal{D}(S)$ is said to be minimal of cyclic type if

$$N^- = (N|_{\mathcal{H}(N)})^-,$$

¹²⁾ For some positive results look at [10, 11, 12] and [22].

where

$$\mathcal{H}(N) = \text{lin} \{N^{*n}x : x \in \mathcal{D}(S), n \geq 0\}.$$

Proposition says that N is minimal of cyclic type. It is not difficult to show, cf. Prop. 3 of [13], that two formally normal extension N_1 and N_2 of S , which are minimal of cyclic type, are $\mathcal{H}$ -equivalent, i.e. there is a unitary operator $U: \mathcal{K}_1 \rightarrow \mathcal{K}_2$ such that

$$Ux = x, \quad x \in \mathcal{H} \quad \text{and} \quad UN_1 = N_2U.$$

If S is subnormal, it may not have any minimal normal extension of cyclic type (cf. [13] and [21]), however, what Cor. 3 of [13] informs us about, if it does at least one, then all its minimal normal extensions of spectral type (they are defined in [13] and always have to exist, cf. Proposition 1(α) of [13]) as well as those minimal of cyclic type are $\mathcal{H}$ -equivalent. Thus we may think of any those equivalent extensions just as a minimal one and say that a subnormal operator S (still having invariant domain) has the *uniqueness extension property* if it has a normal extension being minimal of cyclic type. Going back to our Proposition we can say if the closure N^- of N is normal, S is subnormal and has the uniqueness extension property but not *vice versa*.

Condition (5) is equivalent to positive definiteness of two different form kernels: they lead to two different characterizations of subnormals.

Let $\mathfrak{G}$ be the additive semigroup $\mathbb{N}$ of nonnegative integers. Define K as follows¹³⁾

$$K(m, n, x, y) = \langle S^m x, S^n y \rangle.$$

Then K satisfies (2) with $A(i) = S^i$. Denote by $\mathcal{F}(\mathfrak{G}, \mathcal{E})$ the linear space of all complex valued functions on $\mathfrak{G} \times \mathcal{E}$ and $\mathcal{FL}(\mathfrak{G}, \mathcal{E})$ the linear space of those members f of $\mathcal{F}(\mathfrak{G}, \mathcal{E})$ for which the mapping $x \rightarrow f(s, x)$, $s \in \mathfrak{G}$ are linear.

Theorem 4. *Suppose $SD(S) \subset \mathcal{D}(S)$. Then S is a subnormal operator having the uniqueness extension property if and only if it satisfies (5) and the following holds true: for any $f \in \mathcal{FL}(\mathbb{N}, \mathcal{D}(S))$, $f \neq 0$ such that*

$$f(n+1, Sx) = -f(n, x), \quad n \in \mathbb{N}$$

at least one of the following conditions is satisfied

$$\inf \left\{ \sum_{m,n=0}^M \langle S^m x_n, S^n x_m \rangle \left| \sum_{m=0}^M f(m, x_m) \right|^{-2}; \quad x_0, \dots, x_M \in \mathcal{D}(S), \quad M \in \mathbb{N} \right\} = 0,$$

¹³⁾ A kernel like this is in [4], for instance.

$$\inf\left\{\sum_{m,n=0}^M \langle S^m x_n, S^n x_m \rangle \middle| \sum_{m=0}^M f(m+1, x_m) \right|^{-2}; \quad x_0, \dots, x_M \in \mathcal{D}(S), \quad M \in \mathbb{N}\} = 0.$$

Proof. Let $\mathcal{H}_K$, $\mathcal{D}$, $\mathcal{D}(u)$, Φ_u , $\tilde{\Phi}_u$ and $\Phi(u)$ be as in Theorem 1. Set $N = \tilde{\Phi}_1$. Because

$$\begin{aligned} \|N \sum_{i=0}^N \xi_i K_{n_i, x_i}\| &= \|\tilde{\Phi}_1 \sum_{i=0}^N \xi_i K_{n_i, x_i}\| = \|\Phi(1) \sum_{i=0}^N \xi_i K_{n_i, x_i}\| = \\ &= \left\| \sum_{i=0}^N \xi_i (K_{n_i, x_i})_1 \right\| = \|\tilde{\Phi}_u^* \sum_{i=0}^N \xi_i K_{n_i, x_i}\|^2 = \|N^* \sum_{i=0}^N \xi_i K_{n_i, x_i}\|^2, \end{aligned}$$

N is formally normal. N extends S which follows from $\Phi_u \tilde{V} = VS$. It follows from the construction that N is minimal formally normal extension of S of cyclic type. Thus S is subnormal with the uniqueness extension property if and only if N is normal. Due to Corollary 2, this happens if and only if $f \in \mathcal{D}(\tilde{\Phi}_u^* \Phi(u))$ satisfies (4). However $f \in \mathcal{D}(\tilde{\Phi}_u^* \Phi(u))$ means that $f, f_u, (f_u)^u \in \mathcal{H}_K$. The RKHS test applied to this function leads to the conclusion. ■

Now consider the other possibility. Let $\mathfrak{G} = \mathbb{N} \times \mathbb{N}$ with the involution $(m, n)^* = (n, m)$. Define a form function ω over $(\mathbb{N} \times \mathbb{N}, \mathcal{D}(S))$ as

$$\omega(m, n, x, y) = \omega((m, n), x, y) = \langle S^n x, S^m y \rangle, \quad m, n \in \mathbb{N}, \quad x, y \in \mathcal{D}(S).$$

Let σ be an integer valued solution of

$$\sigma(m+1, n+1) - \sigma(m, n) = \text{odd};$$

call it *admissible*. The result is as follows

Theorem 5. Suppose $S\mathcal{D}(S) \subset \mathcal{D}(S)$. Then S is a subnormal operator having the uniqueness extension property if and only if it satisfies (5) and the following holds true: for any $f \in \mathcal{FL}(\mathbb{Z}, \mathcal{D}(S))$, $f \neq 0$ and any admissible σ at least one of the following conditions is satisfied

$$\inf\left\{\sum_{m,n,p,q=0}^M \langle S^{m+q} x_{m,n}, S^{n+p} x_{pq} \rangle \middle| \sum_{m,n=0}^M (-1)^{\sigma(m,n)} f(m-n, x_{mn}) \right|^2; \quad \{x_{mn}\}_{m,n=0}^M \subset \mathcal{D}(S), \quad M \in \mathbb{N}\} = 0,$$

$$\inf\left\{\sum_{m,n,p,q=0}^M \langle S^{m+q}x_{mn}, S^{n+p}x_{pq} \rangle \left| \sum_{m,n=0}^M (-1)^{\sigma(m+1,n)} f(m-n+1, x_{mn}) \right|^{-2}; \right. \\ \left. \{x_{mn}\}_{m,n=0}^M \subset \mathcal{D}(S), M \in \mathbb{N}\right\} = 0.$$

A few words instead of proof: Set $N = \Phi_{(1,0)}$. Then N is a formally normal extension of S . Because S has invariant domain,

$$\mathcal{H}(N) = \text{lin}\{N^{*n}x : x \in \mathcal{D}(S), n \geq 0\} = \text{lin}\{N^m N^{*n}x : x \in \mathcal{D}(S), m, n \geq 0\}$$

and this implies that the minimality requirement for N is satisfied in this case as well. Then the proof requires merely the arguments of the proof Theorem 4. The only additional information in this case is Theorem 3 where more detailed description of the operators involved is contained.

One may get “weak” versions of conditions of Thms 4 and 5 immediately

Theorem 4’. *Suppose $S\mathcal{D}(S) \subset \mathcal{D}(S)$. Then S is a subnormal operator having the uniqueness extension property if and only if it satisfies (5) and the following holds true: for any $f \in \mathcal{F}(\mathbb{N}, \mathcal{D}(S))$, $f \neq 0$ such that*

$$f(n+1, Sx) = -f(n, x), \quad n \in \mathbb{N}$$

at least one of the following conditions is satisfied

$$\inf\left\{\sum_{m,n=0}^M \langle S^m x_n, S^n x_m \rangle \xi_m \bar{\xi}_n \left| \sum_{m=0}^M f(m, x_m) \xi_m \right|^{-2}; \right. \\ \left. \xi_0, \dots, \xi_M \in \mathbb{C}, x_0, \dots, x_M \in \mathcal{D}(S), M \in \mathbb{N}\right\} = 0, \\ \inf\left\{\sum_{m,n=0}^M \langle S^m x_n, S^n x_m \rangle \xi_m \bar{\xi}_n \left| \sum_{m=0}^M f(m+1, x_m) \xi_m \right|^{-2}; \right. \\ \left. \xi_0, \dots, \xi_M \in \mathbb{C}, x_0, \dots, x_M \in \mathcal{D}(S), M \in \mathbb{N}\right\} = 0.$$

Theorem 5’. *Suppose $S\mathcal{D}(S) \subset \mathcal{D}(S)$. Then S is a subnormal operator having the uniqueness extension property if and only if it satisfies (5) and the following holds true: for any $f \in \mathcal{F}(\mathbb{Z}, \mathcal{D}(S))$, $f \neq 0$ and any admissible σ at least one of the following conditions is satisfied*

$$\inf\left\{\sum_{m,n,p,q=0}^M \langle S^{m+q}x_{mn}, S^{n+p}x_{pq} \rangle \xi_{mn} \bar{\xi}_{pq} \left| \sum_{m,n=0}^M (-1)^{\sigma(m,n)} f(m-n, x_{mn}) \xi_{mn} \right|^{-2}; \right. \\ \left. \{\xi_{mn}\}_{m,n=0}^M \subset \mathbb{C}, \{x_{mn}\}_{m,n=0}^M \subset \mathcal{D}(S), m \in \mathbb{N}\right\} = 0,$$

$$\inf\left\{\sum_{m,n,p,q=0}^M \langle S^{m+q}x_{mn}, S^{n+p}x_{pq} \rangle \xi_{mn} \bar{\xi}_{pq} \mid \sum_{m,n=0}^M (-1)^{\sigma(m+1,n)} \right. \\ \left. f(m-n+1, x_{mn}) \xi_{mn} \right|^{-2}; \quad \{\xi_{mn}\}_{m,n=0}^M \subset \mathbb{C}, \\ \{x_{mn}\}_{m,n=0}^M \subset \mathcal{D}(S), \quad m \in \mathbb{N} \} = 0.$$

The results of this section impact the complex moment problem. We intent to make this more transparent elsewhere.

References

- [1] N. ARONSZAJN, Theory of reproducing kernels, *Trans. Amer. Math. Soc.*, **68** (1950), 337–404.
- [2] E. A. CODDINGTON, Normal extensions of formally normal operators, *Pac. J. Math.*, **10** (1960), 1203–1209.
- [3] Y. KILPI, Lineare normale Transformationen mit einem einfachen Spectrum und das komplexe Momentenproblem, *Ann. Acad. Sci. Fenn. AI*, **176** (1954).
- [4] M. MARTIN and M. PUTINAR, *Lectures on hyponormal operators*, Birkhäuser, Basel, 1989.
- [5] P. MASANI, Dilations as propagators of Hilbertian varieties, *SIAM J. Math. Anal.*, **9** (1978), 414–456.
- [6] G. B. PEDRICK, *Theory of reproducing kernels in Hilbert spaces of vector valued functions*, Univ. of Kansas Tech. Report. #19, Lawrence, 1957.
- [7] F. RIESZ and B. SZ-NAGY, *Lecons d'analyse fonctionnelle*, Académiai Kiadó, Budapest, 1955.
- [8] J. NIECHWIEJ and F. H. SZAFRANIEC, Cosine representations of Bochner type, *Rend. Circ. Mat. Palermo*, **39** (1990), 486–492.
- [9] J. STOCHEL and F. H. SZAFRANIEC, Boundedness of linear and related nonlinear maps Part I – Part II, *Expo. Math.*, **1,2** (1983,1984), 71–73, 283–287.
- [10] J. STOCHEL and F. H. SZAFRANIEC, Bounded vectors and formally normal operators, *Dilation Theory, Toeplitz Operators and Other Topics*, Proceedings, Timisoara-Herculane (Romania), June 7-17, 1982, Operator Theory:Advances and Applications, vol.11, Birkhäuser, Basel (1983), 363–370.
- [12] J. STOCHEL and F. H. SZAFRANIEC, On normal extensions of unbounded operators II, *Acta Sci. Math. (Szeged)*, **53** (1989), 153–177.
- [13] J. STOCHEL and F. H. SZAFRANIEC, On normal extensions of unbounded operators III, Spectral properties, *Publ. RIMS, Kyoto Univ.*, **25** (1989), 105–139.
- [14] F. H. SZAFRANIEC, On the boundedness condition involved in dilation theory, *Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys.*, **24** (1976), 877–881.
- [15] F. H. SZAFRANIEC, A general general dilation theorem, *Bull. Acad. Polon. Sci., Sér. sci. math. astr. et phys.*, **25** (1977), 263–267.

- [16] F. H. SZAFRANIEC, Dilations on involution semigroups, *Proc. Amer. Math. Soc.*, **66** (1977), 30–32.
- [17] F. H. SZAFRANIEC, Apropos of Professor Masani's talk, *Probability Theory on Vector Spaces*, Lecture Notes in Math. vol. 656, Springer-Verlag, Berlin–Heidelberg (1978), 245–249.
- [18] F. H. SZAFRANIEC, Boundedness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.*, **19** (1981), 251–259.
- [19] F. H. SZAFRANIEC, Interpolation and domination by positive definite kernels, *Complex Analysis*, Fifth Romanian Finish Seminar, Part 2, Proceedings, Bucarest (Romania), 1981, Lecture Notes in Math., vol. 1014, Springer-Verlag, Berlin-Heidelberg (1983), 291–295.
- [20] F. H. SZAFRANIEC, Dilations of linear and nonlinear maps, *Functions, Series, Operators*, Proceedings, Budapest (Hungary), 1980, Colloquia Mathematica Societatis János Bolyai, vol. 35, North-Holland, Amsterdam (1983), 1165–1169.
- [21] F. H. SZAFRANIEC, A RKHS of entire functions and its multiplication operator. An explicit example, *Linear Operators in Functions Spaces*, Proceedings, Timisoara (Romania), June 6–16, 1988, Operator Theory: Advances and Applications, vol. 43, Birkhäuser, Basel (1990), 309–312.
- [22] F. H. SZAFRANIEC, Sesquilinear selection of elementary spectral measures and subnormality, *Elementary Operators and Applications*, Proceedings, Blauberger bei Ulm (Deutschland), June 9–12, 1991, World Scientific, Singapore (1992), 243–248.
- [23] B. SZ.-NAGY, *Spectraldarstellung linearer Transformationen des Hilbertschen Raumes*, Springer-Verlag, Berlin, 1942.
- [24] B. SZ.-NAGY, *Prolongement des transformations de l'espace de Hilbert qui sortent de cet espace*, Appendix to [7].
- [25] B. SZ.-NAGY, *Extensions of linear transformations in Hilbert spaces which extend beyond this space* (in Russian), Appendix 1 to the second Russian edition of [7], Izdat. "Mir", Moscow, 1979.
- [26] W. SZYMAŃSKI, The boundedness condition of dilation theory characterizes subnormals and contractions, *Rocky Mountain J. Math.*, **20** (1990), 591–602.

F. H. SZAFRANIEC, Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, PL-30059 Kraków