

Notes on hypercyclic operators

VALENTIN MATACHE

Abstract. The author considers operators T acting on complex separable Banach spaces. The set $\{x, Tx, T^2x, \dots, T^n x, \dots\}$ is called the orbit of x under T . If dense orbits exist T is called *hypercyclic*. If the spectrum of a hypercyclic operator can be represented as the union of two nonvoid, compact, mutually disjoint sets then each of these sets must have nonvoid intersection with the unit circle. No nonzero reducing subspace of a hypercyclic operator T reduces T to a normal, respectively to a compact operator. For Hilbert space contractions A if λA is hypercyclic for some complex λ then A is in $C_0 \setminus C_0$.

1. Introduction

We shall always denote by \mathcal{X} a complex Banach space and by \mathcal{H} a complex Hilbert space. Both spaces are considered separable and infinite-dimensional. When one of them will be supposed to have finite dimension this will be carefully mentioned. We consider only linear, bounded operators acting on \mathcal{X} or \mathcal{H} and say short operators.

An operator T on \mathcal{X} is called *hypercyclic* if there is x in \mathcal{X} such that the set $\{x, Tx, T^2x, \dots, T^n x, \dots\}$ is dense in \mathcal{X} . The set above is called the *orbit* of x . Any vector x in \mathcal{X} having dense orbit is called a *hypercyclic vector* for T .

Some authors call hypercyclic the operators for which any vector but 0 is hypercyclic. Such operators acting on some Banach spaces exist. A remarkable example can be found in [9]. We prefer to call such an operator *everywhere-hypercyclic*. It is not yet known if everywhere-hypercyclic operators acting on infinite-dimensional separable Hilbert spaces exist. If so, obviously such an example would close the celebrated John von Neumann's invariant subspace problem which asks if operators having no non-trivial closed invariant subspaces do exist on infinite-dimensional separable Hilbert spaces.

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In this paper we prove several properties of hypercyclic operators. The most significant results are the following. We show a hypercyclic operator is completely nonnormal and completely noncompact. We deduce that if for a given contraction A there is some complex λ such that λA is hypercyclic then A is a C_0 -contraction in the sense of [11]. However no contraction of class C_0 has the property above.

For the convenience of the reader we shall now state a theorem independently proved by the authors of [6] and [3] which is the main source of examples we know.

Theorem 1.1. *If there are two dense subsets of \mathcal{X} , E and F and some mapping S of F into itself such that $T^n y \rightarrow 0$ for any y in E , $S^n z \rightarrow 0$ and $TSz = z$ for any z in F , then T is hypercyclic.*

Example 1.2. Consider a shift operator A on the Hilbert space \mathcal{H} . For any complex λ such that $|\lambda| > 1$ the operator $T = \lambda A^*$ is hypercyclic.

Indeed one can set E the linear manifold spanned by the vectors in $A^n(\ker A^*)$, $n = 0, 1, 2, 3, \dots$, $S = \lambda^{-1}A$ and F the whole space. This example is taken from [4], the following is from [2].

Example 1.3. Denote by \mathbb{D} the complex open unit disc. Any mapping of \mathbb{D} into itself having the form

$$(1) \quad \phi(z) = \lambda(a - z)(1 - a^*z)^{-1} \quad (z \in \mathbb{D})$$

where λ is a fixed unimodular complex number, a a fixed point in \mathbb{D} and $*$ denotes here complex conjugation, so any mapping of type (1) is called a Möbius transform of \mathbb{D} . If (1) has exactly two fixed points on the unit circle, that is on the boundary of \mathbb{D} , then it is called a hyperbolic Möbius transform. We introduce the operator C_ϕ by

$$(2) \quad C_\phi f = f \circ \phi \quad (f \in H^2)$$

acting on H^2 , the space of all complex functions analytic on \mathbb{D} , having square-summable Taylor coefficients. This is a so-called hyperbolic composition operator. Its boundedness and some spectral properties are in [7]. Theorem 1.1 can be used to show this is a hypercyclic operator. For details we refer to [2]. The main difference between these two examples is that the operator in the latest is invertible. It is plain to see that the inverse of ϕ , ϕ^{-1} is hyperbolic and the composition operator induced by ϕ^{-1} , that is $C_{\phi^{-1}}$ is the inverse of C_ϕ . The inverse of an invertible hypercyclic operator is also hypercyclic. One can find the proof in [1].

2. The main results

We begin by two lemmas which contain the two fundamental ideas behind everything we obtain in this paragraph.

Lemma 2.1. *Let \mathcal{Y} be a Banach space and V a dense-range operator acting from \mathcal{Y} into \mathcal{X} . Let A be a hypercyclic operator on \mathcal{Y} and T an operator on \mathcal{X} . If V intertwines the pair (A, T) , that is if $VA = TV$, then T is hypercyclic too.*

Proof. For any positive integer n , $VA^n = T^nV$. If y in \mathcal{Y} is a hypercyclic vector for A then Vy is a hypercyclic vector for T by the equality above which says V maps the orbit of y with respect to A onto the orbit of Vy with respect to T and by the fact that V has dense range and consequently maps dense sets onto dense sets. ■

Corollary 2.2. *If A is a quasiaffine transform of T in the sense of [11] and A is hypercyclic then T is hypercyclic. Consequently any operator quasi-similar, similar, respectively unitarily equivalent to a hypercyclic operator is hypercyclic.*

Lemma 2.3. *If T is a hypercyclic operator on \mathcal{H} then the compression of T to the orthocomplement of any invariant subspace of T is hypercyclic.*

Proof. Suppose \mathcal{H}_1 is invariant for T , \mathcal{H}_2 is the orthocomplement of \mathcal{H}_1 and T has the following matrix associated to this decomposition of \mathcal{H} into orthogonal direct sum:

$$(3) \quad T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}.$$

Hence

$$(4) \quad T^n = \begin{pmatrix} T_1^n & * \\ 0 & T_2^n \end{pmatrix}.$$

If x is a hypercyclic vector for T and x_2 its projection on \mathcal{H}_2 then by (4) x_2 needs being hypercyclic for T_2 which ends the proof. ■

For Banach space operators we obtain by a similar argument the following particular version of the previous Lemma.

Lemma 2.4. *If T is a hypercyclic operator on \mathcal{X} , P a Banach space projection such that T and P commute then the restriction of T to the range of P is hypercyclic.*

Proof. Denote A the restriction above and let x in \mathcal{X} be a hypercyclic vector for T . Since $A^n Px = T^n Px = PT^n x$ for any positive integer n it follows that Px is a hypercyclic vector for A . ■

The Hilbert space version of the previous Lemma is that any reducing subspace of some hypercyclic operator T on \mathcal{H} reduces T to a hypercyclic operator.

For some operator T on \mathcal{X} we denote $\sigma(T)$ the spectrum of T , $r(T)$ the spectral radius of T and $r_e(T)$ the essential spectral radius of T that is the spectral radius of the image of T in the Calkin algebra.

A pair of nonvoid, compact, mutually disjoint subsets of $\sigma(T)$, (K_1, K_2) is called a Riesz decomposition of $\sigma(T)$ if $\sigma(T)$ is the union of K_1 and K_2 . This is connected to the Riesz–Dunford functional calculus and Riesz’s celebrated theorem which uses this functional calculus to obtain nontrivial hyperinvariant subspaces for operators with disconnected spectrum. We refer the reader to [8] for a detailed treatment of this theory which will be used in the Theorem below.

Theorem 2.5. *If T is a hypercyclic operator on \mathcal{X} then $\sigma(T)$ has nonvoid intersection with the unit circle. If $\sigma(T)$ is disconnected and (K_1, K_2) is a Riesz decomposition of $\sigma(T)$ then both K_1 and K_2 have nonvoid intersection with the unit circle.*

Proof. Suppose $\sigma(T)$ is a subset of \mathbb{D} . Then $r(T) < 1$ and hence $T^n \rightarrow 0$ since $r(T) = \lim \|T^n\|^{1/n}$. Consequently $\{\|T^n\|\}$ is bounded and thus the orbit of any x in \mathcal{X} fails being dense. If we suppose $\sigma(T)$ is outside the closure of \mathbb{D} , then by the spectral mapping theorem $\sigma(T^{-1}) = \sigma(T)^{-1}$ and hence $\sigma(T^{-1})$ is a subset of \mathbb{D} . In this case T^{-1} and thus also T fails being hypercyclic. If T is hypercyclic and $\sigma(T)$ is connected it follows that $\sigma(T)$ has nonvoid intersection with the unit circle. Suppose now $\sigma(T)$ is disconnected, (K_1, K_2) is a Riesz decomposition of $\sigma(T)$. By the Riesz spectral decomposition theory there is an associated pair of Banach space projections (P_1, P_2) both commuting with T . Denote T_1 and T_2 the restrictions of T to the ranges of P_1 and P_2 respectively. The spectra of these restrictions are K_1 and K_2 respectively. By Lemma 2.4 T_1 and T_2 are hypercyclic for which reason both K_1 and K_2 fail being subsets of \mathbb{D} or of the complement of the closure of \mathbb{D} . Suppose now $\sigma(T)$ has void intersection with the unit circle. Since $\sigma(T)$ is disconnected and T is hypercyclic we may set K_1 the intersection of $\sigma(T)$ and \mathbb{D} and K_2 the intersection of $\sigma(T)$ and the complement of the closure of \mathbb{D} to obtain

a Riesz decomposition. By what we have previously observed this is contradictory. We have deduced $\sigma(T)$ has nonvoid intersection with the unit circle. Furthermore, since any term in a Riesz decomposition of the spectrum of a hypercyclic operator is itself the spectrum of another hypercyclic operator it must equally have nonvoid intersection with the unit circle. ■

Remark. The results above made no use of the dimension of \mathcal{X} or \mathcal{H} . They are consequently valid for hypercyclic operators acting on finite-dimensional spaces if any.

Corollary 2.6. *If \mathcal{X} is a nonzero finite-dimensional Banach space then no operator on \mathcal{X} is hypercyclic.*

Proof. According to [10], Chap.I, Theorem 1.21 if \mathbb{C}^n denotes the n -dimensional complex euclidean space with the usual euclidean topology and \mathcal{X} is any n -dimensional complex topological vector space then any linear isomorphism of \mathbb{C}^n onto \mathcal{X} is a homeomorphism. By Corollary 2.2, we deduce it will suffice proving that hypercyclic operators do not exist on finite-dimensional complex euclidean spaces. Suppose the contrary and denote by n the smallest positive integer for which hypercyclic operators exist on \mathbb{C}^n . Obviously $n \geq 2$. Consider T hypercyclic on \mathbb{C}^n . Let \mathcal{M} be a nontrivial invariant subspace of T and deduce by Lemma 2.3 that the compression of T to $\mathbb{C}^n \ominus \mathcal{M}$ is hypercyclic which is absurd since $\mathbb{C}^n \ominus \mathcal{M}$ has dimension less than n . ■

Corollary 2.7. *If T acting on \mathcal{H} is hypercyclic then the compression of T to the orthocomplement of any invariant subspace of T but \mathcal{H} is not compact.*

Proof. If T_2 is the compression above acting on $\mathcal{H}_2 \neq 0$, then by Lemma 2.3 T_2 is hypercyclic and consequently fails being compact. Indeed, since by Corollary 2.6 the dimension of \mathcal{H}_2 is infinite then if T_2 were compact, 0 would be in $\sigma(T_2)$. If 0 is in that case isolated we may set $K_1 = \{0\}$ and K_2 its complement in $\sigma(T_2)$ and obtain a Riesz decomposition with K_1 having no common point with the unit circle. We deduce that T_2 is not hypercyclic. If 0 is not isolated then there is a sequence of isolated eigenvalues that cluster to 0. Take λ in \mathbb{D} an isolated eigenvalue and argue as above to show T_2 is not hypercyclic. ■

We might say any hypercyclic operator is completely noncompact, that is the restriction to any nonzero reducing subspace is not compact.

Corollary 2.8. *If T is a hypercyclic operator on \mathcal{H} and n some positive integer then T^{*n} has void point spectrum.*

Proof. The orthocomplement of any invariant subspace of T is an invariant subspace of T^* . If we take into account Lemma 2.3 and Corollary 2.6 we conclude that a hypercyclic operator has no finite-codimensional invariant subspaces but \mathcal{H} . If v is an eigenvector of T^{*n} then the cyclic subspace $\bigvee_{n=0}^{\infty} T^{*n}v$ is a finite-dimensional invariant subspace of T^* hence T is in that case nonhypercyclic. ■

In the proof of Theorem 2.5 we used the idea that if T is hypercyclic then $r(T) \geq 1$. We are now able to prove more.

Proposition 2.9. *If T is a hypercyclic operator on \mathcal{H} then $r_e(T) \geq 1$.*

Proof. For any compact operator K on \mathcal{H} K^* is also compact and hence $T^{*n} + K^*$ and T^{*n} have the same spectrum except eigenvalues, that is $\sigma(T^{*n}) - \sigma_p(T^{*n}) \subseteq \sigma(T^{*n} + K^*)$. This is Problem 182 in [5]. By Corollary 2.8 we deduce $\sigma(T^{*n}) \subseteq \sigma(T^{*n} + K^*)$ for an arbitrary positive integer n . Taking complex conjugates we obtain $\sigma(T^n) \subseteq \sigma(T^n + K)$ for any compact operator K . Therefore since $r(T) \geq 1$, by the spectral mapping theorem $\sigma(T^n) = (\sigma(T))^n$ we deduce $r(T^n) \geq 1$ and so $1 \leq r(T^n + K) \leq \|T^n + K\|$ for any compact operator K . It follows $1 \leq \|T^n\|_e$ where $\|\cdot\|_e$ denotes the essential norm, that is the norm of the image of the operator in the Calkin algebra. It is now plain that $1 \leq \lim \|T^n\|_e^{1/n} = r_e(T)$. ■

Theorem 2.10. *If T acting on \mathcal{H} is normal then T is not hypercyclic.*

Proof. According to a well known version of the spectral mapping theorem for normal operators (see [8] or [5]), there is a compact subset X of the complex plane, a regular, finite Borel measure μ on X such that T is unitarily equivalent to M_ϕ the multiplication operator on $L^2(X, \mu)$ induced by some essentially bounded ϕ . According to Corollary 2.2 it will suffice to show M_ϕ is not hypercyclic. Denote $B = \{z \in X : |\phi(z)| \leq 1 \text{ a.e.}\}$. If B has positive measure, the multiplication operator induced by χ_B , the characteristic function of B is a nonzero projection. Denote \mathcal{M} its range, that is $\mathcal{M} = \chi_B L^2(X, \mu)$. For any $g = \chi_B f$ in \mathcal{M} we have

$\|M_\phi^n g\| = (\int |\phi(z)|^{2n} |g(z)|^2 d\mu)^{1/2} \leq \|f\|$ for any positive integer n . Consequently the restriction of M_ϕ to the reducing subspace \mathcal{M} is not hypercyclic so, according to Lemma 2.4 M_ϕ is not hypercyclic. Suppose now B has measure 0. Define $B' = \{z \in X : |\phi(z)| > 1 \text{ a.e.}\}$. If B' has positive measure observe $B' = \bigcup_{n=1}^\infty B_n$ where $B_n = \{z \in X : |\phi(z)| > (1+1/n) \text{ a.e.}\}$ and at least for one positive integer m , B_m has positive measure. Consider now the reducing subspace $\mathcal{M} = \chi_B L^2(X, \mu)$ and some nonzero function $g = \chi_{B_m} f$ in \mathcal{M} . Then f is not 0 a.e. on B_m so $\int |g(z)|^2 d\mu > 0$ and $\|M_\phi^n g\| \geq (1 + 1/m)^n (\int |g(z)|^2 d\mu)^{1/2}$. Therefore $\|M_\phi^n g\| \rightarrow \infty$ and consequently the restriction of M_ϕ to \mathcal{M} is not hypercyclic. We deduce that M_ϕ is not hypercyclic. Either B or B' have positive measure for if not then $\mathcal{H} = \{0\}$. ■

Corollary 2.11. *Any hypercyclic operator is completely nonnormal, that is its restriction to any nonzero reducing subspace is not normal.*

We shall investigate in the sequel what conditions are necessary to a contraction A in order that λA be hypercyclic for some complex λ . We begin by

Corollary 2.12. *If for some complex λ , λA is hypercyclic then A is completely nonnormal and consequently completely nonunitary.*

We recall that a contraction A is of class C_0 . in the sense of [11] if $A^n \rightarrow 0$ strongly.

Proposition 2.13. *If for a contraction A , λA is hypercyclic then A is of class C_0 .*

Proof. Denote $\mathcal{M} = \{h \in \mathcal{H} : A^n h \rightarrow 0\}$. \mathcal{M} is obviously an invariant subspace of A . If we suppose A is not of class C_0 . then the orthocomplement of \mathcal{M} is nonzero. Denote $T = \lambda A$ and suppose T is hypercyclic. Denote by T_2 and A_2 respectively the compressions to the orthocomplement of \mathcal{M} of T and A respectively. Obviously $T_2 = \lambda A_2$. According to Lemma 2.3 T_2 is hypercyclic hence T_2 has dense range and consequently so does A_2 since obviously $\lambda \neq 0$. On the other hand A_2 is of class C_1 . that is if $h \neq 0$ then $A_2^n h \not\rightarrow 0$ (see [11], Chap.II, proof of Theorem 4.1). According to [11], Chap.II, Proposition 3.5 there is some unitary R_* and some quasiaffinity Y such that $R_* Y = Y A_2$ and consequently $\lambda R_* Y = Y T_2$. By Corollary 2.2 it follows λR_* is hypercyclic which is absurd since λR_* is normal. ■

Here is an application of this result. Suppose ϕ is a hyperbolic Möbius transform. We can always write $\phi(z) = (az + c^*) / (cz + a^*)$ with $\text{Re } a > 0$ and $|a|^2 - |c|^2 =$

1. Denote then $K = (\operatorname{Re} a + (|c|^2 - (\operatorname{Im} a)^2)^{1/2} / (\operatorname{Re} a - (|c|^2 - (\operatorname{Im} a)^2)^{1/2})$. It is known (see [7]) that the spectrum of the hyperbolic composition operator C_ϕ is the annulus centered at 0 of outer radius $K^{1/2}$ and inner radius $K^{-1/2}$. We denote it by σ . It is equally shown in [7] that its interior consists only of eigenvalues.

Application 1. *If $K^{1/2} = |a| + |c|$ then the point spectrum of C_ϕ is exactly the interior of σ and consequently the approximate point spectrum of C_ϕ coincides with σ .*

Proof. According to [7] $\|C_\phi\| = (1 + |\phi(0)|)^{1/2}(1 - |\phi(0)|)^{-1/2}$ hence $\|C_\phi\| = |a| + |c| = r(C_\phi) = K^{1/2}$. Denote $A = C_\phi / \|C_\phi\|$ which is a contraction of class C_0 , and has consequently no eigenvalues on the unit circle. Hence C_ϕ has no eigenvalues on the circle of center 0 and radius $K^{1/2}$. The same is true for the circle of center 0 and radius $K^{-1/2}$, for if C_ϕ had an eigenvalue on this circle then its inverse C_ϕ^{-1} which is also hypercyclic would have an eigenvalue on the circle of center 0 and radius $K^{1/2}$. But obviously $|\phi^{-1}(0)| = |c|/|a|$, $\|C_{\phi^{-1}}\| = \|C_\phi\| = |a| + |c| = r(C_{\phi^{-1}}) = K^{1/2}$. The fact that $C_{\phi^{-1}}$ has the same spectral radius as C_ϕ is a simple application of the spectral mapping theorem, that is $\sigma(C_{\phi^{-1}}) = \sigma(C_\phi)^{-1}$. Now by the same argument we have already used for C_ϕ it follows that the assumption that $C_{\phi^{-1}}$ had eigenvalues on the circle of radius $K^{1/2}$ is absurd. Hence the point spectrum is the interior of σ and since the boundary of the spectrum is always contained in the approximate point spectrum the desired result is clear. ■

We recall that a completely nonunitary contraction A acting on \mathcal{H} is of class C_0 in the sense of [11] if there is some inner function u such that $u(A) = 0$ in the sense of the Nagy–Foias functional calculus for contractions. C_0 contains C_0 (see once more [11]).

Proposition 2.14. *For any contraction A of class C_0 there is no complex λ such that λA be hypercyclic.*

Proof. Suppose contrary that $T = \lambda A$ is hypercyclic for some complex λ . Obviously $|\lambda| > 1$ for if not, T is a contraction, has uniformly bounded iterates and fails being hypercyclic. Denote by m the minimal inner annihilating function of A . By a well known theorem in [11], $\sigma(A)$ consists of the zeros of m in \mathbb{D} if any and the complement in the unit circle of the union of open arcs on which m is analytic. Denote these two subsets of $\sigma(A)$ by σ_1 and σ_2 respectively. If σ_1 is void then $\sigma(T)$ has void intersection with the unit circle and T is not hypercyclic according to

Theorem 2.5. We admit then σ_1 is nonvoid. If σ_2 is also nonvoid and the distance between σ_1 and σ_2 is strictly positive then set $K_1 = \lambda\sigma_1$ and $K_2 = \lambda\sigma_2$ and obtain a Riesz decomposition of $\sigma(T)$ with K_2 having void intersection with the unit circle. Once more by Theorem 2.5 T is not hypercyclic. We admit then that the zeros of m cluster towards the unit circle. Since m is analytic on \mathbb{D} any of these zeros is an isolated point in $\sigma(A)$. We may chose w a zero of m such that the distance between w and σ_2 be small enough that $|\lambda w| > 1$ holds. Set now $K_1 = \{\lambda w\}$ and $K_2 = \sigma(T) - K_1$ to obtain a Riesz decomposition of $\sigma(T)$ such that K_1 and the unit circle be disjoint. Again T is not hypercyclic. Finally if σ_2 is void then m has finitely many zeros in \mathbb{D} and is analytic on the whole unit circle. Consequently m is in that case a Blaschke product with finitely many factors. We deduce A is an algebraic operator that is, there is some nonzero polynomial p having complex coefficients such that $p(A) = 0$. T is also algebraic and hence for any h in \mathcal{H} the cyclic subspace generated by h is finite-dimensional for which reason the orbit of h fails being dense. ■

We give a second application of these results. Example 1.2 shows that for any scalar λ with $|\lambda| > 1$, λA^* is hypercyclic where A^* is the adjoint of some shift operator A on \mathcal{H} . A^* is obviously a C_{01} -contraction in the sense of [11]. Therefore the following result might seam a little striking. Nevertheless we can prove.

Application 2. Denote by A the shift operator of multiplicity 1 represented on H^2 as multiplication by z . For any nonconstant inner function u denote B the restriction of A^* to the invariant subspace $(uH^2)^\perp$. For any complex λ , λB is not hypercyclic.

Proof. According to [1], Chap.III, Proposition 4.3 the restriction above is a contraction of class C_0 having minimal inner annihilating function u . ■

We acknowledge having learned from [4] of the existence of the following result proved in [6]. Any connected component of the spectrum of a hypercyclic operator has nonvoid intersection with the unit circle. Unfortunately [6] is an unpublished doctoral thesis. All our efforts to get a copy remained successless. It is in order mentioning this here since our Theorem 2.5 can be easily deduced from this result. We have obtained Theorem 2.5 independently.

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V. MATACHE, Department of Mathematics, University of Timișoara, Bul. Vasile Pârvan nr.4, 1900-Timișoara, România