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Semimodularity and irreducible elements

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Abstract. We generalize results concerning the relation between semimodularity and irreducible elements.

In [7] Walendziak generalizes some results obtained by Stern [6] for lattices of finite length to lower continuous strongly dually atomic lattices. In fact, he characterizes semimodularity defined as in [2] by some properties of join-irreducible elements. It is apparent that under this artificial condition join-irreducible elements coincide with completely join-irreducible elements. What is more, Theorem 2 in [7] holds in reality for all atomistic lattices. The aim of this paper is to find out what is really essential.

Recall that for each element a of a lattice L the set $\{b \in L \mid b < a\}$ has the supremum, which we will denote by a'. An element a is said to be *completely join-irreducible* if $a' < a$. The set of all completely join-irreducible elements will be denoted by $J(L)$. A lattice is *semimodular* if $a \wedge b \prec a$ implies $b \prec a \vee b$ for each $a, b \in L$. A lattice is *atomistic* if each of its elements is the join of a set of atoms. The set of all atoms will be denoted by $At(L)$.

Notice that the following conditions are equivalent for each $u \in J(L)$ and $b \in L$:

$u \not\leq b$ & $u' \leq b$;

$u \wedge b \prec u$;

$u \wedge b = u'$.

We put $u^* := \{b \in L \mid u \wedge b = u'\}$.

Further, the following conditions are equivalent:

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For all $u, v \in J(L)$ and $b \in L$ such that $v \leq b \vee u$ and $v \not\leq b \vee u'$ we have $u \leq b \vee v \vee u'$;
for all $u, v \in J(L)$ and $b \in u^*$ such that $v \leq b \vee u$ and $v \not\leq b$ we have $u \leq b \vee v$.

Definition. We say that L has the *exchange property for completely join-irreducible elements* if it satisfies the preceding conditions. We say that a lattice L is *semibased* if for each $a, b, c \in L$ such that $a \wedge b \prec a$ and $b < c \leq a \vee b$ we have a completely join-irreducible element u such that $u \leq c$, $u \not\leq b$ and if moreover $b \prec a$, then $u' \leq b$. We say that L is *semiatomistic* if for each $a, b, c \in L$ such that $a \wedge b \prec a$ and $b < c \leq a \vee b$ we have an atom u such that $u \leq c$ and $u \not\leq b$.

Lemma. *In a semiatomistic lattice, $J(L) = At(L)$.*

Proof. An atom is completely join-irreducible in every lattice. Conversely, if L is semiatomistic and $u \in J(L)$, then there exists an atom w such that $w \leq u$ and $w \not\leq u'$. Hence $w \vee u' = u$ and therefore $w = u$. ∎

Lemma.
(1) *Atomistic lattices are semiatomistic.*
(2) *Semiatomistic lattices are semibased.*

The proof is obvious. Lower continuous strongly dually atomic lattices investigated in [7] are semibased.

Proposition 1. *The following conditions are equivalent for a semibased lattice L:*
(i) *L is semimodular;*
(ii) *L has the exchange property for completely join-irreducible elements;*
(iii) *for each $u \in J(L)$ and $b \in u^*$ we have $b \prec b \vee u$.*

Proof. (i)$\Longrightarrow$(ii) holds in every lattice.

(ii)$\Longrightarrow$(iii): Let $u \in J(L)$, $b \in u^*$. Consider $b < c \leq b \vee u$. Since L is semibased, there exists a completely join-irreducible element $v \leq c$ such that $v \not\leq b$. By (ii), $b \vee u = b \vee v \leq c$. Hence $b \prec b \vee u$.

(iii)$\Longrightarrow$(i): Let $a \wedge b \prec a$. Since L is semibased, there exists $u \in J(L)$ such that $u \leq a$, $u \not\leq a \wedge b$ and $u' \leq a \wedge b$. Hence $u' = b \wedge u$. By (iii), $b \prec b \vee u$. Now $b \vee u = b \vee (a \wedge b) \vee u = b \vee a$. ∎

The next proposition is a particular case of the preceding one.

Proposition 2. *The following conditions are equivalent for a semiatomistic lattice L:*

(i) *L is semimodular;*

(ii) *L has the Steinitz–Mac Lane exchange property, i.e. for all $u, v \in At(L)$ and $b \in L$ such that $v \leq b \vee u$ and $v \not\leq b$ we have $u \leq b \vee v$;*

(iii) *for each $u \in At(L)$ and $b \in L$ such that $b \wedge u = 0$ we have $b \prec b \vee u$.*

The preceding results can be generalized to ordered sets as follows. In [3] it is shown that completely meet-irreducible elements in ordered sets are precisely relatively maximal elements. Dually, completely join-irreducible elements are precisely relatively minimal elements. Recall that an element a is said to be *relatively maximal with respect to* b if it is a maximal element in $\{c \mid b \not\leq c\}$. It is said to be *relatively maximal* if it is relatively maximal with respect to an element.

Lemma. *Let L be a lattice, $a, b \in L$. Then $a \wedge b \prec a$ if and only if a is relatively minimal with respect to b in the dual principal ideal generated by $a \wedge b$.*

Proof. Let $a \wedge b \prec a$. Then $a \not\leq b$ and $a \wedge b \leq c < a$ implies $c \leq b$. Hence a is relatively minimal with respect to b in the dual principal ideal generated by $a \wedge b$. Let conversely a be relatively minimal with respect to b in the dual principal ideal generated by $a \wedge b$. Then $a \wedge b < c \leq a$ implies $c = a$, and hence $a \wedge b \prec a$. ∎

Notice that an element a of an ordered set is relatively minimal with respect to b in a dual Frink ideal (cf. [1]) if and only if it is relatively minimal with respect to b in $U(L(a, b))$ where U denotes the set of all upper bounds and L the set of all lower bounds.

Definition. We say that an ordered set P is *semibased* if whenever a is relatively minimal with respect to b in $U(L(a, b))$, then there exists an element $u \leq a$ relatively minimal with respect to b such that $U(u, b) = U(a, b)$ and to each $c \in L(U(a, b))$ such that $b < c$ there exists a relatively minimal element v such that $v \leq c$, $v \not\leq b$. We say that an ordered set P is *semimodular* if whenever a is relatively minimal with respect to b in $U(L(a, b))$, then b is relatively maximal with respect to a in $L(U(a, b))$.

Proposition 3. *The following conditions are equivalent for a semibased ordered set P:*

(i) *P is semimodular;*

(ii) *If u is relatively minimal with respect to b and $v \in L(U(u,b))$ is relatively minimal such that $v \not\leq b$, then $U(b,v) = U(b,u)$ in $L(U(u,b))$;*

(iii) *If u is relatively minimal with respect to b, then b is relatively maximal with respect to u in $L(U(u,b))$.*

Proof. Implication (i)$\Longrightarrow$(ii) holds for every ordered set. Let u be relatively minimal with respect to b and $v \in L(U(u,b))$ relatively minimal such that $v \not\leq b$. Then b is relatively maximal with respect to u in $L(U(u,b))$. Hence $c \in U(b,v) \cap L(U(u,b))$ yields $c \in U(b,u)$.

(ii)$\Longrightarrow$(iii): Suppose (ii) is satisfied. Let u be relatively minimal with respect to b and $c \in L(U(u,b))$ such that $b < c$. Since P is semibased, there exists a relatively minimal element $v \leq c$ such that $v \not\leq b$. Then $c \in U(b,v) \cap L(U(u,b))$, and hence $c \in U(u,b)$.

(iii)$\Longrightarrow$(i): Let a be relatively minimal with respect to b in $U(L(a,b))$. Since P is semibased, there exists an element $u \leq a$ relatively minimal with respect to b such that $U(u,b) = U(a,b)$. Now b is relatively maximal with respect to u in $L(U(u,b))$. Hence for each $c \in L(U(a,b)) = L(U(u,b))$ such that $b < c$ we obtain $c \in U(u,b) = U(a,b)$. Therefore b is relatively maximal with respect to a in $L(U(a,b))$. ∎

Another approach to this subject is that developed in [5]. Every ordered set P can be investigated as a doubly dense generating subset in its characteristic lattice $G(P)$, that is the lattice generated by it in its Dedekind–MacNeille completion.

Let P be an ordered set and $P \subseteq Q \subseteq G(P)$, $\{[u', u] \mid u \in J(G(P))\} \subseteq R \subseteq <$ where the relation $<$ is considered in $G(P)$. We define $a \prec_Q b$ to mean that $a < b$ and $Q \cap (a,b) = \emptyset$. For $u \in J(G(P))$ we put $u^* := \{b \in P \mid b \wedge u = u'\}$. P is said to be (Q,R)-*semibased* if for each $a, b \in P$ such that $a \wedge b \; R \; a$ we have an element $u \in J(G(P))$ with $b \in u^*$, $(a \wedge b) \vee u = a$ and for each $c \in Q \cap (b, a \vee b\rangle$ an element $v \in J(G(P))$ with $v \leq c$, $v \not\leq b$. P is said to be (Q,R)-*semimodular* if for each $a, b \in P$ such that $a \wedge b \; R \; a$ we have $b \prec_Q a \vee b$.

Theorem. *Let P be (Q,R)-semibased. The following conditions are equivalent:*

(i) *P is (Q,R)-semimodular;*

(ii) *for each $u, v \in J(G(P))$ and $b \in u^*$ such that $v \leq b \vee u$, $v \not\leq b$ we have $\langle b \vee v, b \vee u) \cap Q = \emptyset$;*

(iii) *for each $u \in J(G(P))$ and $b \in u^*$ we have $b \prec_Q b \vee u$.*

Proof. (i)$\Longrightarrow$(ii): Let $u, v \in J(G(P))$, $b \in u^*$, $v \leq b \vee u$, $v \not\leq b$, $b \vee v \leq c < b \vee u$. Since $u \in J(G(P)) \subseteq P$ and $b \wedge u = u' \; R \; u$, it follows that $b \prec_Q b \vee u$. Thus $c \notin Q$.

(ii)$\Longrightarrow$(iii): Let $u \in J(G(P))$ and $b \in u^*$. Then $u \in P$, $b \in P$, $b \wedge u = u' \; R \; u$ and hence for each $c \in Q \cap (b, b \vee u\rangle$ we have an element $v \in J(G(P))$ such that $v \leq c$, $v \not\leq b$. Since $Q \cap \langle b \vee v, b \vee u) = \emptyset$, we obtain $c = b \vee u$. Thus $b \prec_Q b \vee u$.

(iii)$\Longrightarrow$(i): Let $a, b \in P$ such that $a \wedge b \; R \; a$. There exists $u \in J(G(P))$ such that $b \in u^*$, $(a \wedge b) \vee u = a$. Then $b \vee u = b \vee (a \wedge b) \vee u = a \vee b$. This yields that $b \prec_Q a \vee b$ by assumption.

∎

In [5] it is proved that whenever an ordered set P is a doubly dense generating subset of a lattice L, then L is canonically isomorphic to $G(P)$. The covering relation in the subsequent corollaries is considered in L.

Corollary. *Let P be a doubly dense generating subset of a lattice L such that for each $a, b \in P$ with $a \wedge b \prec a$ we have an element $u \in J(L)$ with $u \leq a$, $b \in u^*$, and for each $c \in L$ with $b < c \leq a \vee b$ we have an element $v \in J(L)$ with $v \leq c$, $v \not\leq b$. The following conditions are equivalent:*

(i) *For each $a, b \in P$ with $a \wedge b \prec a$ we have $b \prec a \vee b$;*
(ii) *for each $u, v \in J(L)$ and $b \in u^*$ with $v \leq u \vee b$, $v \not\leq b$ we have $u \leq v \vee b$;*
(iii) *for each $u \in J(L)$ and $b \in u^*$ we have $b \prec u \vee b$.*

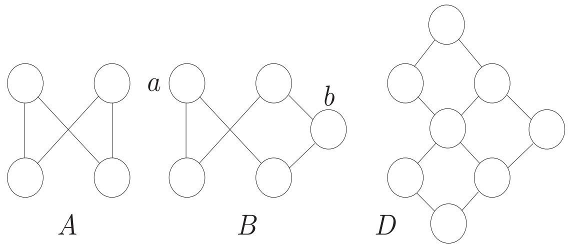
Corollary. *Let P be a doubly dense generating subset of a lattice L such that for each $a, b \in P$ with $a \wedge b \prec a$ we have an element $u \in J(L)$ with $u \leq a$, $b \in u^*$, and for each $c \in P$ with $b < c \leq a \vee b$ we have an element $v \in J(L)$ with $v \leq c$, $v \not\leq b$. The following conditions are equivalent:*

(i) *For each $a, b, c \in P$ with $a \wedge b \prec a$, $b < c \leq a \vee b$ we have $c = a \vee b$;*
(ii) *for each $u, v \in J(L)$, $b \in u^*$ and $c \in P$ with $v \leq u \vee b$, $v \not\leq b$, $v \vee b \leq c \leq u \vee b$ we have $c = u \vee b$;*
(iii) *for each $u \in J(L)$, $b \in u^*$ and $c \in P$ with $b < c \leq u \vee b$ we have $c = u \vee b$.*

In fact, Propositions 1 and 3 are also corollaries of the Theorem: the former for $Q := L$, $R := \prec$, the latter for $Q := P$, $R := \prec_P$.

Concluding remarks

Semibased lattices generalize lattices of finite length on the one hand and chains on the other. Semiatomistic lattices of finite length are obviously atomistic. The real interval $\langle 0,1\rangle$ can serve as an example of a semiatomistic lattice which is not atomistic, and the ordinal sum $\langle 0,1\rangle \oplus \langle 0,1\rangle$ as an example of a semibased lattice which is not semiatomistic. Semibased and semimodular ordered sets generalize semibased and semimodular lattices respectively. The ordered set A visualized below is semimodular whereas the ordered set B is not.



Nevertheless, B is distributive since it can be obtained by removing some doubly reducible elements from the distributive lattice D (cf. [4]). Every distributive ordered set P is $(G(P), -\!\!<)$-semimodular.

There are surely other possibilities how to define semimodular ordered sets.

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