

Smoothness of Green's functions and density of sets

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Abstract. We investigate local properties of the Green function of the complement of a compact set $E \subset [0, 1]$ with respect to the extended complex plane. We extend results of V. Andrievskii, L. Carleson and V. Totik which claim that the Green function satisfies the $1/2$ -Hölder condition locally at the origin if and only if the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0, 1]$. We give an integral estimate on the density in terms of the Green function and extend the results to the case $E \subset [-1, 1]$.

1. Definitions and main results

Behavior of Green's functions around boundary points is a fundamental question of harmonic analysis, which has applications in different areas such as smoothness properties of solutions to Dirichlet problems or bounds for polynomials and polynomial inequalities.

Let $E \subset [0, 1]$ be a compact set with positive (logarithmic) capacity $\text{cap}(E) > 0$. We consider E as a set in the complex plane $\mathbb{C}$ and use notions of potential theory in the plane (see [6], [5]).

Let $\Omega := \overline{\mathbb{C}} \setminus E$, where $\overline{\mathbb{C}} := \{\infty\} \cup \mathbb{C}$ is the extended complex plane. Denote by $g_\Omega(z) = g_\Omega(z, \infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞ . In what

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follows we assume that 0 is a regular point of E , i.e., $g_\Omega(z)$ extends continuously to 0 and $g_\Omega(0) = 0$.

The monotonicity of the Green function yields

$$g_\Omega(z) \geq g_{\mathbb{C} \setminus [0,1]}(z), \quad z \in \mathbb{C} \setminus [0,1],$$

that is, if E has the “highest density” at 0, then g_Ω has the “highest smoothness” at the origin. In particular

$$g_\Omega(-r) \geq g_{\mathbb{C} \setminus [0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of E whose Green function has the “highest smoothness” at 0, that is, of E conforming to the following condition

$$g_\Omega(z) \leq C|z|^{1/2}, \quad z \in \mathbb{C},$$

which is known to be the same as

$$(1.1) \quad g_\Omega(-r) \leq Cr^{1/2}, \quad 0 < r < 1$$

(c.f. [1, Theorem 3.6]). Various sufficient conditions for (1.1) in terms of metric properties of E are stated in [7], where the reader can also find further references.

There are compact sets $E \subset [0, 1]$ of linear Lebesgue measure 0 with property (1.1) (see e.g. [7, Corollary 5.2]), hence (1.1) may hold, though the set E is not dense at 0 in terms of linear measure. On the contrary, V. Andrievskii [2] proved that if E satisfies (1.1) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrarily close to the density of $[0, 1]$ in that neighborhood, i.e. (1.1) implies

$$(1.2) \quad \lim_{r \rightarrow 0} \frac{\text{cap}(E \cap [0, r])}{r} = \frac{1}{4}.$$

Recall that $\text{cap}(I) = |I|/4$ for any interval I , where $|I|$ denotes the length (Lebesgue measure) of I .

For $0 < \varepsilon < 1/2$ we set (see [4])

$$(1.3) \quad E_\varepsilon(t) = (E \cap [0, t]) \cup [0, \varepsilon t] \cup [(1 - \varepsilon)t, t].$$

Theorem 1. *For any $\varepsilon > 0$*

$$(1.4) \quad \int_r^1 \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < C_0 \frac{g_\Omega(-r)}{\sqrt{r}}$$

where C_0 is independent of r .

L. Carleson and V. Totik [4] have characterized the optimal smoothness in terms of a Wiener type condition. They proved

Theorem 2. *Let $\varepsilon < 1/3$. Then E satisfies (1.1) if and only if*

$$\sum_k \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(2^{-k}))}{2^{-k}} \right) < \infty.$$

This theorem plays the same role for Lip 1/2 smoothness as Wiener's theorem for continuity. The proof of Theorem 2 in [4], due to L. Carleson, was based on Poisson's formula. There is an alternative approach: using the technique of balayage; and with it we prove the following variant of Theorem 2.

Theorem 3. *Let $\varepsilon < 1/2$. Then E satisfies (1.1) if and only if*

$$(1.5) \quad \int_0^1 \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < \infty.$$

Andrievskii's theorem is a consequence of Theorem 3 (see Lemma 1).

Andrievskii constructed a regular compact set $E \subset [0, 1]$ such that

$$\lim_{r \rightarrow 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0 < \varepsilon < \frac{1}{2}$$

holds and

$$(1.6) \quad \liminf_{r \rightarrow 0} \frac{\text{cap}(E \cap [0, r])}{r} = 0.$$

Furthermore he proved that conversely, (1.2) does not imply (1.1).

The method used in the proofs of Theorems 1 and 3 can be applied to the case $E \subset [-1, 1]$ as well (c.f. [4, Theorem 1.11]). In this case

$$g_\Omega(ir) \geq g_{\mathbb{C} \setminus [-1, 1]}(ir) > \frac{r}{2}, \quad 0 < r < 1,$$

therefore in this case the optimal smoothness for Green functions is Hölder 1 and we are interested in the sets E satisfying

$$g_{\Omega}(z) \leq C|z|, \quad 0 < |z| < 1.$$

This is equivalent to

$$(1.7) \quad g_{\Omega}(ir) \leq Cr, \quad 0 < r < 1$$

because $g_{\Omega}(x + iy)$ is monotone in y (c.f. the reasoning before (6.5) in the Proof of Theorem 5). The highest smoothness of the Green function at the origin (Lipschitz condition) is again equivalent to the highest density at 0. Namely, let $E \subset [-1, 1]$ and set $E_{\varepsilon}(t)$ as in (1.3) and

$$E_{\varepsilon}(-t) = \left(E \cap [-t, 0]\right) \cup [-t, (1 - \varepsilon)(-t)] \cup [-\varepsilon t, 0].$$

Theorem 4. *If $E \subseteq [-1, 1]$ and $\varepsilon > 0$ then*

$$(1.8) \quad \int_r^1 \left(\frac{1}{4} - \frac{\text{cap}(E_{\varepsilon}(t))}{t} \right) \frac{1}{t} dt < C_0 \frac{g_{\Omega}(ir)}{r}$$

The same is true for $E_{\varepsilon}(-t)$.

Theorem 5. *Let $\varepsilon < 1/2$. Then E satisfies*

$$(1.9) \quad g_{\Omega}(z) \leq C|z|, \quad 0 < |z| < 1,$$

if and only if (1.5) holds for $E_{\varepsilon}(t)$ and $E_{\varepsilon}(-t)$.

This is a variant of [4, Theorem 1.11].

Corollary 1. *If E satisfies (1.9) then*

$$(1.10) \quad \lim_{r \rightarrow 0} \frac{\text{cap}(E \cap [-r, r])}{r} = \frac{1}{2}.$$

Corollary 2. (c.f. [4, Corollary 1.12]) g_Ω is Hölder 1 continuous at 0 if and only if both $g_{\mathbb{C} \setminus (E \cap [0,1])}$ and $g_{\mathbb{C} \setminus (E \cap [-1,0])}$ are Hölder 1/2 continuous there.

2. Notations

We shall use $c, c_0, c_1, c_2, \dots, C, C_0, C_1, C_2, \dots$ and $d_1, d_2, \dots$ to denote positive constants. These constants may be either absolute or they may depend on E depending on the context. We may use the same symbol for different constants if this does not lead to confusion.

$|F|$ denotes the linear Lebesgue measure of a measurable subset $F \subset \mathbb{R}$ of the real line $\mathbb{R}$.

$\mathbb{D} := \{z : |z| < 1\}$ is the unit disk, $\mathbb{T} = \partial\mathbb{D}$ is the unit circle and for $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$ let

$$[z_1, z_2] := \{tz_2 + (1-t)z_1 : 0 \leq t \leq 1\}$$

be the interval between these points.

For the notions of logarithmic potential theory see e.g. [5] or [6]. In what follows μ_E denotes the equilibrium measure of E ,

$$U^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t)$$

the logarithmic potential of the measure ν , $g_G(z, a)$ the Green function of the domain G with pole at a , $\omega(x, H, G)$ the harmonic measure in G corresponding to the set $H \subseteq \partial G$. We shall frequently use the relation

$$(2.1) \quad g_{\mathbb{C} \setminus E}(z) = \log \frac{1}{\text{cap}(E)} - U^{\mu_E}(z), \quad z \in \mathbb{C} \setminus E$$

valid for any compact set E of positive capacity.

Let G be a domain with compact boundary and with $\text{cap}(\partial G) > 0$, and let ν be a measure supported on $\overline{G}$. We shall need the concept of balayage (or sweeping) of ν out of G (sometimes we say balayage onto ∂G), see e.g. [6, Sec. II.4]. It is the unique measure $\overline{\nu}$ supported on ∂G with the property that $\nu(\mathbb{C}) = \overline{\nu}(\mathbb{C})$ (total mass is the same) and

$$(2.2) \quad U^{\overline{\nu}}(z) = U^\nu(z) + \text{const}$$

for $z \in \partial G$ with the exception of a set of capacity 0. For regular G the exceptional set is empty. If G is bounded, then the constant is 0 ([6, Ch. II, Theorem 4.1]), and if G is unbounded, then it is ([6, Ch. II, Theorem 4.4])

$$(2.3) \quad \text{const} = \int_G g_G(a, \infty) d\nu(a).$$

We shall use the notation $\text{Bal}(\nu, G)$ for the balayage measure $\bar{\nu}$.

There is a connection between harmonic and balayage measures: if $K \subseteq \partial G$ are compact sets, then for $x \in G$ the equality

$$(2.4) \quad \text{Bal}(\delta_x, G)(K) = \omega(x, K, G)$$

holds, where δ_x denotes the point mass (Dirac measure) placed at the point x (see e.g. [6, Appendix A3, (3.3)]). Therefore, in what follows we shall interchangeably use the harmonic measure and balayage notations.

We shall also use Harnack's inequality: if u is a positive harmonic function in the unit disk and $|z| < 1$, then

$$(2.5) \quad \frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0).$$

It follows from this (see e.g. [3]) that if K is a compact subset of G , then there is a constant c such that for all positive harmonic functions u on G

$$cu(x) \leq u(y) \leq \frac{1}{c}u(x)$$

for all x and y in K .

3. Proof of Theorem 1

We divide the proof into three steps.

Step I. First we are going to verify the following: Let $I_j = [a_j, b_j]$, $j \in \mathbb{N}$ be disjoint closed subintervals of $(0, 1]$ such that $b_j \leq C_1 |I_j|$, $j \in \mathbb{N}$ for some C_1 , and for $\varepsilon > 0$ set

$$(3.1) \quad F_j = (I_j \cap E) \cup [a_j, a_j + (\varepsilon/2)|I_j|] \cup [b_j - (\varepsilon/2)|I_j|, b_j].$$

Then

$$(3.2) \quad \sum_{j: I_j \subseteq [r, 1]} \left(\frac{1}{4} - \frac{\text{cap}(F_j)}{|I_j|} \right) < c_0 \frac{g_\Omega(-r)}{\sqrt{r}}.$$

For the proof first of all notice that

$$(3.3) \quad \mu_{E \cup [0, r]}([0, r]) \leq C_2 g_\Omega(-r), \quad 0 < r < 1,$$

for some $C_2 > 0$ (recall that $\mu_{E \cup [0, r]}$ denotes the equilibrium measure of $E \cup [0, r]$). This is immediate, since (see (2.1))

$$\begin{aligned} g_\Omega(-r) &\geq g_{\mathbb{C} \setminus (E \cup [0, r])}(-r) = \log \frac{1}{\text{cap}(E \cup [0, r])} - U^{\mu_{E \cup [0, r]}}(-r) \\ &= U^{\mu_{E \cup [0, r]}}(0) - U^{\mu_{E \cup [0, r]}}(-r) = \int \log \frac{t+r}{t} d\mu_{E \cup [0, r]}(t) \\ &\geq (\log 2) \int_0^r d\mu_{E \cup [0, r]}(t) = (\log 2) \mu_{E \cup [0, r]}([0, r]). \end{aligned}$$

Let D_j resp. C_j be the open disk, resp. circle with diameter I_j , let J_j be the middle third part of the arc $\partial D_j \cap \{\Im z > 0\}$. If $I = (a, b)$ we use the notation $I(\varepsilon) = (a + \varepsilon(b-a), b - \varepsilon(b-a))$. Taking balayage of some measure supported in D_j onto $E \cup [0, r]$ can be done in two steps: first take balayage onto $\partial(D_j \setminus (E \cup [0, r]))$, and then onto $E \cup [0, r]$ (see Figure 1). Hence for $a \in I_j(\varepsilon/2) = [a_j + (\varepsilon/2)|I_j|, b_j - (\varepsilon/2)|I_j|]$, $I_j \subset [r, 1]$

$$\begin{aligned} &\text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \\ &= \int_{C_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) d\text{Bal}(\delta_a, D_j \setminus E)(b). \end{aligned}$$

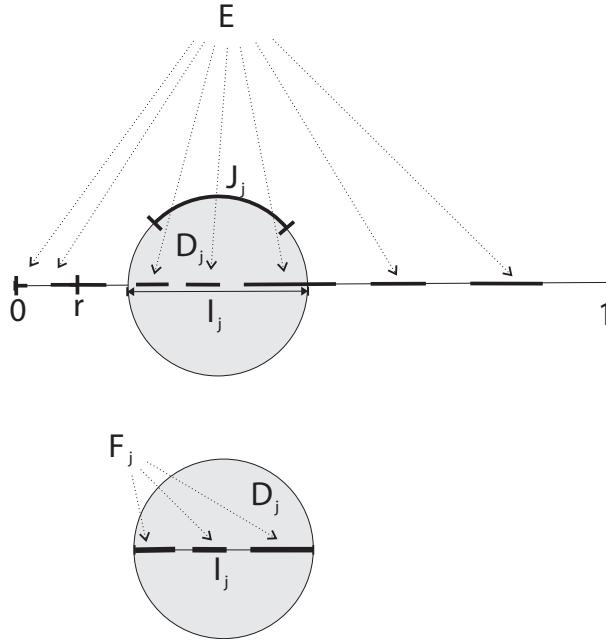


Figure 1: the disk D_j and the set F_j

Next we use that $\mu_{E \cup [0, r]}$ is the balayage of $\mu_{[0, 1]}$ onto $E \cup [0, r]$ ([6, Theorem IV.1.6, (e)]), and so

$$\begin{aligned}
 \mu_{E \cup [0, r]}([0, r]) &= \text{Bal}(\mu_{[0, 1]}, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \\
 &\geq \text{Bal}\left(\mu_{[0, 1]}|_{[r, 1] \setminus E}, \mathbb{C} \setminus (E \cup [0, r])\right)([0, r]) \\
 &= \int_{[r, 1] \setminus E} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) d\mu_{[0, 1]}(a) \\
 (3.4) \quad &\geq \sum_{j: I_j \subseteq [r, 1]} \int_{I_j} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) d\mu_{[0, 1]}(a) \\
 &= \sum_{j: I_j \subseteq [r, 1]} S_j,
 \end{aligned}$$

and here

$$\begin{aligned}
 S_j &\geq \int_{I_j(\varepsilon/2)} \int_{C_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) d \text{Bal}(\delta_a, D_j \setminus E)(b) d\mu_{[0, 1]}(a) \\
 &\geq \left(\inf_{b \in J_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \right) \int_{I_j(\varepsilon/2)} \text{Bal}(\delta_a, D_j \setminus E)(J_j) d\mu_{[0, 1]}(a).
 \end{aligned}$$

Denote by $2C_j$, resp. $2D_j$ what we obtain from C_j resp. D_j by enlarging them twice from their center. Then $2D_j$ contains in its interior the interval I_j . For $a \in I_j(\varepsilon/2)$ Lemma 2 gives

$$\begin{aligned} \text{Bal}(\delta_a, D_j \setminus E)(J_j) &\geq \text{Bal}(\delta_a, D_j \setminus F_j)(J_j) \geq c_{\varepsilon/2} \text{Bal}(\delta_a, D_j \setminus F_j)(C_j) \\ &\geq c_{\varepsilon/2} \text{Bal}(\delta_a, 2D_j \setminus F_j)(2C_j) = c_{\varepsilon/2} \omega(a, 2C_j, 2D_j \setminus F_j). \end{aligned}$$

Since $g_{\mathbb{C} \setminus F_j}(z) \sim 1 = \omega(z, 2C_j, 2D_j \setminus F_j)$ for $z \in 2C_j$, and both $g_{\mathbb{C} \setminus F_j}(z)$ and $\omega(z, 2C_j, 2D_j \setminus F_j)$ are harmonic in $2D_j \setminus F_j$ and vanish on F_j , these functions are comparable throughout $2D_j \setminus F_j$. Therefore the preceding estimate yields a constant $c > 0$ such that for $a \in I_j(\varepsilon/2)$ we have

$$\text{Bal}(\delta_a, D_j \setminus E)(J_j) \geq c g_{\mathbb{C} \setminus F_j}(a).$$

Therefore we can continue the inequality for S_j as

$$(3.5) \quad S_j \geq c \left(\inf_{b \in J_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \right) \int_{I_j(\varepsilon/2)} g_{\mathbb{C} \setminus F_j}(a) d\mu_{[0,1]}(a).$$

Here

$$d\mu_{[0,1]}(a) = \frac{1}{\pi \sqrt{a(1-a)}} da,$$

and hence for $a \in I_j = [a_j, b_j]$ we have

$$(3.6) \quad \frac{1}{\pi \sqrt{a(1-a)}} \geq \frac{1}{\pi \sqrt{b_j}} \geq \frac{1}{4C_1 \sqrt{|I_j|}}$$

by the assumption $b_j \leq C_1 |I_j|$. If

$$\theta_j = \frac{1}{4} - \frac{\text{cap}(F_j)}{|I_j|} = \frac{1}{|I_j|} (\text{cap}(I_j) - \text{cap}(F_j))$$

then

$$(3.7) \quad \theta_j \sim \log \frac{\text{cap}(I_j)}{\text{cap}(F_j)} = \int_{I_j} g_{\mathbb{C} \setminus F_j}(t) d\mu_{I_j}(t) \sim \frac{1}{|I_j|} \int_{I_j} g_{\mathbb{C} \setminus F_j}(t) dt,$$

where the equality is known (see e.g. [4, (2.7)]) and the last relation is true because the integrals are actually integrals over $[a_j + (\varepsilon/2)|I_j|, b_j - (\varepsilon/2)|I_j|] \setminus E$ and $d\mu_{I_j}(t) = 1/\left(\pi \sqrt{(t-a_j)(b_j-t)}\right) \sim 1/|I_j| dt$ there. Thus we obtain

$$(3.8) \quad S_j \geq c_1 \theta_j \sqrt{|I_j|} \left(\inf_{b \in J_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \right).$$

For $b \notin E \cup [0, r]$ the quantity

$$\text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) = \omega(b, [0, r], \mathbb{C} \setminus (E \cup [0, r]))$$

is a nonnegative harmonic function of b , hence by Harnack's inequality we have for $b \in J_j$ and $d_j = |I_j|$ the inequality

$$\text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \geq c_2 \text{Bal}(\delta_{-d_j}, \mathbb{C} \setminus (E \cup [0, r]))([0, r])$$

with some absolute constant $c_2 > 0$ because $\text{dist}(J_j, 0) \sim \text{dist}(J_j, [0, 1]) \sim |I_j| = d_j$. By [6, Ch. II, (4.47)] we have

$$\begin{aligned} & \text{Bal}(\delta_{-d_j}, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \\ (3.9) \quad & \geq \text{Bal}(\delta_{-d_j}, \mathbb{C} \setminus [0, 1])([0, r]) = \frac{1}{\pi} \int_0^r \frac{\sqrt{d_j} \sqrt{1+d_j}}{\sqrt{t(1-t)}(t+d_j)} dt \\ & \geq \frac{1}{\pi} \frac{\sqrt{r}}{\sqrt{d_j}} = \frac{1}{\pi} \frac{\sqrt{r}}{\sqrt{|I_j|}}. \end{aligned}$$

This, the previous inequality, (3.8) and (3.4) give

$$\mu_{E \cup [0, r]}([0, r]) \geq c_3 \sqrt{r} \sum_{j: I_j \subseteq [r, 1]} \theta_j,$$

which together with (3.3) proves (3.2).

Step II. Let $E \subseteq [0, 1]$ be compact and for $\varepsilon > 0$, $0 < t < 1$ set

$$(3.10) \quad E_\varepsilon^*(t) = \left(E \cap [\varepsilon t/2, t] \right) \cup [\varepsilon t/2, \varepsilon t] \cup [(1 - \varepsilon/2)t, t].$$

Then for $0 < q < 1$

$$(3.11) \quad \sum_{m: q^m > \frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon^*(q^m))}{q^m(1 - \varepsilon/2)} \right) < c_0 \frac{g_\Omega(-r)}{\sqrt{r}},$$

where c_0 depends only on ε and q .

To prove this let the integer M be so large that $q^M < \varepsilon/2$. Clearly, it is sufficient to show that for each $l = 1, \dots, M$ the sum for the subsequence $m = jM + l$, $j \in \mathbb{N}$ satisfies

$$\sum_{j: q^{jM+l} > \frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon^*(q^{jM+l}))}{q^{jM+l}(1 - \varepsilon/2)} \right) < c_l \frac{g_\Omega(-r)}{\sqrt{r}}.$$

But this immediately follows from the result proved in Part 1, since the intervals $I_j = [\varepsilon q^{jM+l}/2, q^{jM+l}]$, $j \in \mathbb{N}$ are pairwise disjoint and the set F_j defined in (3.1) for these intervals is contained in $E_\varepsilon^*(q^{jM+l})$.

Step III. Finally, we complete the proof of Theorem 1. Let $\varepsilon > 0$ and $0 < u < 1$. If $u \leq t \leq u(1 - \varepsilon/2)/(1 - \varepsilon)$, then for the sets (1.3) and (3.10) the relation $E_\varepsilon(t) \cap [\varepsilon u/2, u] \supseteq E_\varepsilon^*(u)$ holds, and so

$$(3.12) \quad \frac{\text{cap}(E_\varepsilon^*(u))}{u(1 - \varepsilon/2)} \leq \frac{\text{cap}(E_\varepsilon(t) \cap ([\varepsilon u/2, u]))}{u(1 - \varepsilon/2)}.$$

But $E_\varepsilon(t) = [0, \varepsilon u/2] \cup (E_\varepsilon(u) \cap [\varepsilon u/2, u]) \cup [u, t]$, i.e. $E_\varepsilon(t)$ is obtained from $E_\varepsilon(u) \cap [\varepsilon u/2, u]$ by attaching one-one intervals to the right and to the left. Therefore, we can apply Lemma 3 below ((7.10), twice) to conclude

$$\frac{\text{cap}(E_\varepsilon(u) \cap ([\varepsilon u/2, u]))}{u(1 - \varepsilon/2)} \leq \frac{\text{cap}(E_\varepsilon(t))}{t},$$

which, together with (3.12), gives

$$(3.13) \quad \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \leq \frac{1}{4} - \frac{\text{cap}(E_\varepsilon^*(u))}{u(1 - \varepsilon/2)}.$$

This is true for all $u \leq t \leq u(1 - \varepsilon/2)/(1 - \varepsilon)$, therefore if we divide both sides by t and integrate with respect to t over the interval $[u, u(1 - \varepsilon/2)/(1 - \varepsilon)]$ then we obtain with $q = (1 - \varepsilon)/(1 - \varepsilon/2)$

$$(3.14) \quad \int_u^{u/q} \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt \leq \left(\log \frac{1 - \varepsilon/2}{1 - \varepsilon} \right) \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon^*(u))}{u(1 - \varepsilon/2)} \right).$$

Let k be the largest integer for which $q^k > 2r/\varepsilon$. Summing up (3.14) for $u = q, q^2, q^3, \dots, q^k$ and making use of (3.11) we obtain

$$\int_{q^k}^1 \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \frac{1}{t} dt < C_3 \frac{g_\Omega(-r)}{\sqrt{r}}.$$

Since

$$q^k \leq \frac{2r}{\varepsilon} \frac{1}{q} = \frac{2r}{\varepsilon} \frac{1 - \frac{\varepsilon}{2}}{1 - \varepsilon} \leq \frac{4r}{\varepsilon},$$

we can change the limit of the integral to $\frac{4r}{\varepsilon}$. Then, changing $\frac{4r}{\varepsilon}$ for r we can use Harnack's inequality to obtain

$$(3.15) \quad g_\Omega\left(-\frac{\varepsilon r}{4}\right) \leq C_4 g_\Omega(-r),$$

where C_4 depends only on ε . This completes the proof of Theorem 1. ■

4. Proof of Theorem 3

It follows from Theorem 1 that (1.1) implies (1.5). Therefore we only need to show that the converse is true. We divide the proof into two steps.

Step I. First we are going to verify the following: If $0 < q < 1$ and $\varepsilon < q/(q+1)$ then

$$(4.1) \quad \sum_{m=1}^{\infty} \left(\frac{1}{4} - \frac{\text{cap}(E_{\varepsilon}(q^m))}{q^m} \right) < \infty$$

implies (1.1).

For the proof let

$$\theta_m^* = \frac{1}{4} - \frac{\text{cap}(E_{\varepsilon}(q^m))}{q^m}$$

and suppose $\sum \theta_m^* < \infty$. Let $\eta > 0$ be chosen below, choose M so that $\sum_{m \geq M} \theta_m^* < \eta$, and let $\tilde{E} = E \cup [\varepsilon q^{-M}, 1]$. By considering Green's functions $g_{\mathbb{C} \setminus E}$ and $g_{\mathbb{C} \setminus \tilde{E}}$ on a small circle about the origin not intersecting $\tilde{E}$, we can see that the Hölder $1/2$ property at the origin for $g_{\mathbb{C} \setminus E}$ is the same as for $g_{\mathbb{C} \setminus \tilde{E}}$, therefore we may assume that $E = \tilde{E}$, i.e. $\sum_m \theta_m^* < \eta$ and (for sufficiently large M) $[(1-\alpha)q/2, 1] \subseteq E$, where $\alpha = q/(q+1)$.

It is sufficient to show that $\mu_E([0, \delta]) = O(\sqrt{\delta})$. In fact, then

$$\begin{aligned} g_{\Omega}(-r) &= g_{\Omega}(-r) - g_{\Omega}(0) = \int \log \frac{r+t}{t} d\mu_E(t) \\ &= \sum_{m=0}^{\infty} \int_{r/2^{m+1}}^{r/2^m} + \sum_{m: r \leq 2^m r < 1} \int_{2^m r}^{2^{m+1} r} \\ &= \sum_{m=0}^{\infty} O\left((m+1)\sqrt{\frac{r}{2^m}}\right) + \sum_{m=0}^{\infty} O\left(\frac{1}{2^m} \sqrt{2^m r}\right) = O(\sqrt{r}). \end{aligned}$$

This time let $I_m = [0, q^m]$ and C_m^* resp. D_m^* be the circle resp. disk with diameter $I_m(\varepsilon) = [\varepsilon q^m, (1-\varepsilon)q^m]$ (c.f. Figure 2). For $a \in I_m(\alpha) = [\alpha q^m, (1-\alpha)q^m]$ we obtain from Lemma 2

$$\omega(a, C_m^*, D_m^* \setminus E) \leq d\omega(a, J_m^*, D_m^* \setminus E),$$

where J_m^* is the middle third part of the arc $C_m^* \cap \{\Im z > 0\}$ (see Lemma 2) and $d = 1/c_{2(\alpha-\varepsilon)/(1-2\varepsilon)}$. Let $E_m = E_{\varepsilon}(q^m)$. Since for $z \in J_m^*$ we have $g_{\mathbb{C} \setminus E_m}(z) \sim 1$

and $g_{\mathbb{C} \setminus E_m}(z) \geq 0$ for $z \in C_m^* \setminus J_m^*$, it follows from the maximum principle and the previous inequality that with some constant d_1

$$(4.2) \quad \omega(a, C_m^*, D_m^* \setminus E) \leq d_1 g_{\mathbb{C} \setminus E_m}(a) \quad \text{for any } a \in I_m(\alpha).$$

Let n be large, $\mu_n = \mu_{E \cup I_n}$ the equilibrium measure of $E \cup I_n = E \cup [0, q^n]$, and define M_n as $\mu_n([0, q^n]) = M_n q^{n/2}$. Since $\mu_E([0, q^n]) \leq \mu_n([0, q^n])$ (note that μ_E is the balayage of μ_n out of Ω), it is sufficient to show that $M_n = O(1)$. μ_n is obtained by taking the balayage of $\mu_{[0,1]}$ onto $E \cup I_n$, hence

$$(4.3) \quad M_n q^{n/2} = \mu_{[0,1]}(I_n) + \int_{[q^n, 1] \setminus E} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup I_n))(I_n) d\mu_{[0,1]}(a).$$

Set p such that $q^n < \alpha q^p < q^{n-1}$. Then $\cup_{m=1}^\infty I_m(\alpha) \supset (0, (1-\alpha)q]$ and since $[(1-\alpha)q/2, 1] \subseteq E$, the last integral can be written as

$$(4.4) \quad \int_{q^n}^{\alpha q^p} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup I_n))(I_n) d\mu_{[0,1]}(a) + \\ + \sum_{m=1}^p \int_{I_m(\alpha)} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup I_n))(I_n) d\mu_{[0,1]}(a) = S_0^* + \sum_{m=1}^p S_m^*.$$

Clearly

$$S_0^* + \mu_{[0,1]}(I_n) \leq \mu_{[0,1]}([0, q^{n-1}]) \leq q^{n/2}.$$

Therefore, if we can show that with some constant d_2 depending only on $\varepsilon < q/(q+1)$ we have $S_m^* \leq d_2 \theta_m^* M_n q^{n/2}$, then (4.3) and (4.4) give

$$M_n q^{n/2} \leq q^{n/2} + d_2 M_n q^{n/2} \sum_{m=1}^p \theta_m^* \leq q^{n/2} + d_2 M_n q^{n/2} \eta,$$

and for $\eta < 1/2d_2$ this yields $M_n \leq 2$.

Thus, it is left to prove $S_m^* \leq d_2 \theta_m^* M_n q^{n/2}$. Since the balayage measure $\text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup I_n))$ can be obtained by taking first balayage out of $D_m^* \setminus E$ and then taking the balayage of the so obtained measure out of $\mathbb{C} \setminus (E \cup I_n)$, we have the formula

$$\text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup I_n))(I_n) = \int_{C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup I_n))(I_n) d\text{Bal}(\delta_a, D_m^* \setminus E)(b).$$

Here

$$\text{Bal}(\delta_a, D_m^* \setminus E)(C_m^*) = \omega(a, C_m^*, D_m^* \setminus E),$$

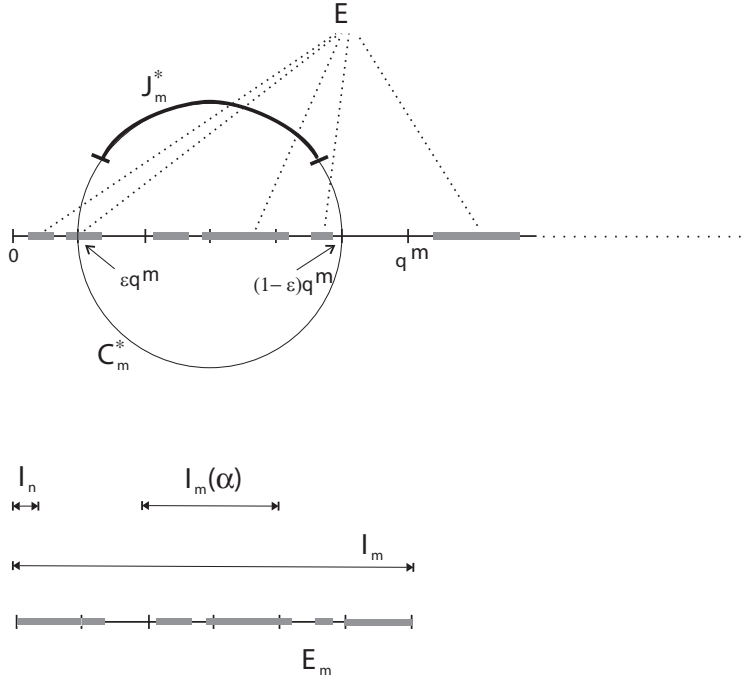


Figure 2: the disk D_m^* and the set E_m

hence

$$\begin{aligned}
 (4.5) \quad S_m^* &= \int_{I_m(\alpha)} \int_{C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup I_n))(I_n) d\text{Bal}(\delta_a, D_m^* \setminus E)(b) d\mu_{[0,1]}(a) \\
 &\leq \left(\sup_{b \in C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup I_n))(I_n) \right) \int_{I_m(\alpha)} \omega(a, C_m^*, D_m^* \setminus E) d\mu_{[0,1]}(a).
 \end{aligned}$$

Since for $a \in I_m(\alpha)$ the equilibrium measure $\mu_{[0,1]}(a)$ is given by the density $1/(\pi\sqrt{a(1-a)}) \sim q^{-m/2}$, we obtain from (4.2)

$$S_m^* \leq \left(\sup_{b \in C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup I_n))(I_n) \right) \int_{I_m(\alpha)} g_{\mathbb{C} \setminus E_m}(a) d_3 q^{-m/2} da.$$

Now, just like in (3.7)

$$\theta_m^* \sim q^{-m} \int_{I_m} g_{\mathbb{C} \setminus E_m}(t) dt,$$

therefore the last integral is at most $d_4 q^{m/2} \theta_m^*$, hence it is left to show

$$\sup_{b \in C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup I_n))(I_n) \leq d_5 \text{Bal}(\delta_\infty, \mathbb{C} \setminus (E \cup I_n))(I_n) q^{-m/2},$$

because $\text{Bal}(\delta_\infty, \mathbb{C} \setminus (E \cup I_n))$ is the equilibrium measure of $E \cup I_n$ and

$$\text{Bal}(\delta_\infty, \mathbb{C} \setminus (E \cup I_n))(I_n) = \mu_n(I_n) = M_n q^{n/2}.$$

We estimate the harmonic measure $\text{Bal}(\delta_z, \mathbb{C} \setminus (E \cup I_n))(I_n) = \omega(z, I_n, \overline{\mathbb{C}} \setminus (E \cup I_n))$ for $z \in C_m^*$ by taking the conformal map

$$w(z) = \left(q^{-n-1} z - 1 \right) - \sqrt{\left(q^{-n-1} z - 1 \right)^2 - 1}$$

of $\overline{\mathbb{C}} \setminus I_n$ onto the unit disk $\mathbb{D}$. This maps I_n onto $\mathbb{T}$, $E \setminus I_n$ into a subset E^* of $[q^{n+2}, 1]$, the point ∞ into the origin and the circle C_m^* into a closed curve γ such that all points of γ are of distance $\sim q^{n-m}$ from 0. Thus, there is a constant d_6 such that γ lies inside the circle $\mathbb{T}_{d_6 q^{n-m}}$ of radius $d_6 q^{n-m}$ about the origin. Now for $b \in C_m^*$ we obtain from the maximum principle

$$\begin{aligned} \omega(b, I_n, \overline{\mathbb{C}} \setminus (E \cup I_n)) &= \omega(w(b), \mathbb{T}, \mathbb{D} \setminus E^*) \\ &\leq \sup_{w \in \mathbb{T}_{d_6 q^{n-m}}} \omega(w, \mathbb{T}, \mathbb{D} \setminus E^*) = \omega(-d_6 q^{n-m}, \mathbb{T}, \mathbb{D} \setminus E^*), \end{aligned}$$

where the last equality follows from the solution to Milloux' problem (see [1, Section 3.3]). Hence, it is enough to prove

$$(4.6) \quad \omega(-d_6 q^{n-m}, \mathbb{T}, \mathbb{D} \setminus E^*) \leq d_7 q^{-m/2} \omega(0, \mathbb{T}, \mathbb{D} \setminus E^*).$$

Map now $\mathbb{D}$ onto the exterior of $[-1, 1]$ by $v(w) = (w + 1/w)/2$, and then the exterior of $[-1, q^{-n-2} - 1]$ onto $\mathbb{D}$ by

$$u(v) = \left(q^{n+1}(v + 1) - 1 \right) - \sqrt{\left(q^{n+1}(v + 1) - 1 \right)^2 - 1}.$$

Under these mappings the point $-d_6 q^{n-m}$ is mapped into $-1 + r_m$ with $r_m \sim q^{m/2}$, while 0 is mapped into 0, furthermore with $z = u(v(w)) = h(w)$ the function $\omega(h^{-1}(z), \mathbb{T}, \mathbb{D} \setminus E^*)$ is harmonic in $\mathbb{D}$ (note that the image of E^* under $w \mapsto h(w)$ is part of the unit circle). Hence Harnack's inequality gives

$$\omega(h^{-1}(-1 + r_m), \mathbb{T}, \mathbb{D} \setminus E^*) \leq d_7 q^{-m/2} \omega(h^{-1}(0), \mathbb{T}, \mathbb{D} \setminus E^*),$$

and this is (4.6).

Step II. Let $\varepsilon' < 1/2$ and suppose (1.5) holds with ε' in place of ε . Then (4.1) holds for $\varepsilon' < \varepsilon < q/(q+1)$ ($q < 1$).

Let $0 < u < 1$. If $u(1-\varepsilon)/(1-\varepsilon') \leq t \leq u$, then for the sets (1.3) and $E_\varepsilon(u)$ the relation $E'_\varepsilon(t) \subset E_\varepsilon(u) \cap [0, t]$ holds, and so

$$(4.7) \quad \frac{\text{cap}(E_{\varepsilon'}(t))}{t} \leq \frac{\text{cap}(E_\varepsilon(u) \cap [0, t])}{t}.$$

Since $E_\varepsilon(u) = (E_\varepsilon(u) \cap [0, t]) \cup [t, u]$, applying Lemma 3 (like in the Proof of Theorem 1) we can conclude

$$\frac{\text{cap}(E_{\varepsilon'}(t))}{t} \leq \frac{\text{cap}(E_\varepsilon(u))}{u},$$

hence

$$(4.8) \quad \frac{1}{4} - \frac{\text{cap}(E_{\varepsilon'}(t))}{t} \geq \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(u))}{u}.$$

Dividing both sides by t and integrating with respect to t over the interval $[u(1-\varepsilon)/(1-\varepsilon'), u]$ we obtain with $q = (1-\varepsilon)/(1-\varepsilon')$

$$(4.9) \quad \int_{qu}^u \left(\frac{1}{4} - \frac{\text{cap}(E_{\varepsilon'}(t))}{t} \right) \frac{1}{t} dt \geq \left(\log \frac{1-\varepsilon'}{1-\varepsilon} \right) \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(u))}{u} \right).$$

Summing up for $u = 1, q, q^2, q^3, \dots, q^m$ and making use of (1.5) we obtain

$$\sum_{m=1}^{\infty} \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(q^m))}{q^m} \right) < \infty,$$

and the proof of Theorem 3 is complete. ■

5. Proof of Theorem 4

First of all notice that in the proof of Theorem 1 we used the fact that $E \subset [0, 1]$ only in Step I. Actually, we used it at two main steps: proving (3.8) (using the equilibrium measure of $[0, 1]$) and establishing (3.9). Therefore, we will only mention the steps where the proof differs from that of Theorem 1.

We are going to use the notations of Step I. Instead of (3.3) now we have

$$(5.1) \quad \mu_{E \cup [0, r]}([0, r]) \leq C_2 g_\Omega(ir), \quad 0 < r < 1.$$

Indeed,

$$\begin{aligned} g_\Omega(ir) &\geq g_{\mathbb{C} \setminus (E \cup [0, r])}(ir) = U^{\mu_{E \cup [0, r]}}(0) - U^{\mu_{E \cup [0, r]}}(ir) = \int \log \left| \frac{ir - t}{t} \right| d\mu_{E \cup [0, r]}(t) \\ &\geq \log \sqrt{2} \int_0^r d\mu_{E \cup [0, r]}(t) = \left(\log \sqrt{2} \right) \mu_{E \cup [0, r]}([0, r]). \end{aligned}$$

Replacing $\mu_{[0, 1]}$ by $\mu_{[-1, 1]}$ in the argument before (3.5) we have (c.f. (3.5))

$$(5.2) \quad S_j \geq c \left(\inf_{b \in J_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \right) \int_{I_j(\varepsilon)} g_{\mathbb{C} \setminus F_j}(a) d\mu_{[-1, 1]}(a).$$

Now (c.f. the proof of (3.8)) we have

$$(5.3) \quad S_j \geq c_1 \theta_j |I_j| \left(\inf_{b \in J_j} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \right),$$

since

$$d\mu_{[-1, 1]}(t) = \frac{1}{\pi \sqrt{(1+t)(1-t)}} dt$$

and (c.f. (3.6))

$$\frac{1}{\pi \sqrt{(1+t)(1-t)}} \geq \frac{1}{\pi}.$$

In (3.9) we used δ_{-d_j} . Now, since $-d_j$ may be in E , let us change it for id_j . By Harnack's inequality we have for $b \in J_j$

$$\begin{aligned} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) &\geq c_1 \text{Bal}(\delta_{id_j}, \mathbb{C} \setminus (E \cup [0, r]))([0, r]) \\ &\geq c_1 \text{Bal}(\delta_{id_j}, \mathbb{C} \setminus [-1, 1])([0, r]) \\ &= c_1 \omega(id_j, [0, r], \mathbb{C} \setminus [-1, 1]). \end{aligned} \quad (5.4)$$

Applying the transformation $\varphi(z) = z - \sqrt{z^2 - 1}$ and using ([6, Ch. II, (4.8)]) we have

$$\begin{aligned} &\omega(id_j, [0, r], \mathbb{C} \setminus [-1, 1]) \\ &= \omega\left(i\left(d_j - \sqrt{1 + d_j^2}\right), A, D\right) \\ &= \frac{1}{2\pi} \left(\int_{\arccos r}^{\pi/2} P\left(\zeta, i\left(d_j - \sqrt{1 + d_j^2}\right)\right) dt + \int_{3\pi/2}^{2\pi - \arccos r} P\left(\zeta, i\left(d_j - \sqrt{1 + d_j^2}\right)\right) dt \right), \end{aligned}$$

where $\zeta = e^{it}$, P is the Poisson kernel and A is the intersection $\mathbb{T} \cap \{z : 0 \leq \Re(z) \leq r\}$ consisting of two arcs on the unit circle. Thus,

$$\begin{aligned} & \omega(id_j, [0, r], \mathbb{C} \setminus [-1, 1]) \\ & \geq \frac{1}{2\pi} \int_{\frac{3\pi}{2}}^{2\pi - \arccos r} \frac{1 - (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})}{1 - 2(\sqrt{1+d_j^2} - d_j) \cos\left(t + \frac{\pi}{2}\right) + (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})} dt \\ & \geq \frac{1}{\pi} r \frac{d_j(\sqrt{1+d_j^2} - d_j)}{1 - 2(\sqrt{1+d_j^2} - d_j)\sqrt{1-r^2} + (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})}. \end{aligned}$$

Assuming $d_j \geq r$ we get

$$\begin{aligned} & \omega(id_j, [0, r], \mathbb{C} \setminus [-1, 1]) \\ & \geq \frac{r}{\pi} \frac{d_j(\sqrt{1+d_j^2} - d_j)}{1 - 2(\sqrt{1+d_j^2} - d_j)\sqrt{1-d_j^2} + (1 + 2d_j^2 - 2d_j\sqrt{1+d_j^2})} \\ & = \frac{r}{2\pi} \frac{d_j(\sqrt{1+d_j^2} - d_j)}{1 - (\sqrt{1+d_j^2} - d_j)(\sqrt{1-d_j^2} + d_j)} = \frac{r}{2\pi} f(d_j), \end{aligned}$$

where

$$f(x) = \frac{x(\sqrt{1+x^2} - x)}{1 - (\sqrt{1+x^2} - x)(\sqrt{1-x^2} + x)} = \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{2x}.$$

$xf(x)$ is monotone decreasing on $[0, 1]$, hence

$$f(x) \geq \frac{f(1)}{x} = \frac{\sqrt{2}}{2} \left(\frac{1}{x}\right),$$

which gives

$$\omega(id_j, [0, r], \mathbb{C} \setminus [-1, 1]) \geq \frac{\sqrt{2}}{4\pi} \left(\frac{r}{d_j}\right).$$

This, (5.4) and (5.3) give

$$S_j \geq c_3 \theta_j r$$

for $d_j \geq r$, $I_j \subset [r, 1]$, which proves

$$\sum_{j: I_j \subseteq [r, 1], |I_j| \geq r} \left(\frac{1}{4} - \frac{\text{cap}(F_j)}{|I_j|} \right) < c_0 \frac{g_\Omega(ir)}{r},$$

and the proof of Step I is complete.

The requirement $|I_j| \geq r$ in the summation doesn't affect the proofs in Steps II and III, because if $q^m > \frac{2r}{\varepsilon}$, then $r \leq q^m(1 - \varepsilon/2) = |I_j|$. Finally, in (3.15) we can change the arguments for $i\varepsilon r/4$ and ir respectively. ■

6. Proof of Theorem 5 and Corollary 1

Again, we will only mention the steps where the proof differs from that of Theorem 3. We only need to show that if $E_\varepsilon(t)$ and $E_\varepsilon(-t)$ satisfy (1.5), then (1.7) holds. We need some changes in Step I.

It suffices to show that $\mu_E([- \delta, \delta]) = O(\delta)$. This time let $\mu_n = \mu_{E \cup [-q^n, q^n]}$ the equilibrium measure of $E \cup [-q^n, q^n]$, and define M_n as $\mu_n([-q^n, q^n]) = M_n q^n$. Again it is sufficient to show that $M_n = O(1)$. μ_n is obtained by taking the balayage of $\mu_{[-1,1]}$ onto $E \cup [-q^n, q^n]$, hence

$$(6.1) \quad \begin{aligned} M_n q^n &= \mu_{[-1,1]}([-q^n, q^n]) + \\ &+ \int_{[q^n, 1] \setminus E} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [-q^n, q^n]))([-q^n, q^n]) d\mu_{[-1,1]}(a) + \\ &+ \int_{[-1, -q^n] \setminus E} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [-q^n, q^n]))([-q^n, q^n]) d\mu_{[-1,1]}(a). \end{aligned}$$

Now

$$\mu_{[-1,1]}([-q^{n-1}, q^{n-1}]) \leq c q^n$$

and we can write the two integrals as sums like before. We will only deal with the first integral, the other one can be handled similarly. Let α , $I_m(\alpha)$ and θ_m^* be as in the proof of Theorem 3. It suffices to show that with some constant d_2 depending only on ε we have $S_m^* \leq d_2 \theta_m^* M_n q^n$, where

$$S_m^* = \int_{I_m(\alpha)} \text{Bal}(\delta_a, \mathbb{C} \setminus (E \cup [-s^n, s^n]))([-q^n, q^n]) d\mu_{[-1,1]}(a)$$

and $I_m(\alpha) = [\alpha q^m, (1 - \alpha)q^m]$.

The inequality in (4.5) and the equations before remain valid if we change I_n for $[-q^n, q^n]$ and $[0, 1]$ for $[-1, 1]$ (and C_m^* and D_m^* are the circle and disk with diameter I_m). This time for $a \in I_m(\alpha)$ the density of the equilibrium measure $\mu_{[-1,1]}(a)$ is $1/(\pi \sqrt{(a+1)(1-a)}) \sim 1$, hence it is left to show

$$\begin{aligned} &\sup_{b \in C_m^*} \text{Bal}(\delta_b, \mathbb{C} \setminus (E \cup [-q^n, q^n]))([-q^n, q^n]) \\ &\leq d_5 \text{Bal}(\delta_\infty, \mathbb{C} \setminus (E \cup [-q^n, q^n]))([-q^n, q^n]) q^{-m}. \end{aligned}$$

The conformal map

$$w(z) = q^{-n}z - \sqrt{(q^{-n}z)^2 - 1}$$

takes $\overline{\mathbb{C}} \setminus [-q^n, q^n]$ onto the unit disk $\mathbb{D}$, $[-q^n, q^n]$ onto $\mathbb{T}$, $E \setminus [-q^n, q^n]$ into a subset E^* of $[-1, -q^{n+2}] \cup [q^{n+2}, 1]$, the point ∞ into the origin and the circle C_m^* into a closed curve γ such that all points of γ are of distance $\sim q^{n-m}$ from 0. Hence, it is enough to prove

$$(6.2) \quad \sup_{w \in \mathbb{T}_{d_6 q^{n-m}}} \omega(w, \mathbb{T}, \mathbb{D} \setminus E^*) \leq d_7 q^{-m} \omega(0, \mathbb{T}, \mathbb{D} \setminus E^*).$$

Let $\delta < \varepsilon_0$, where ε_0 is defined in Lemma 4. Set $E^{**} = [-1 + \delta, 1 - \delta] \cap E^*$. The image of E^{**} under w^{-1} is $E \cap \left([-1, -s^{-n}(2 - \delta + \delta^2)/(2 - \delta)] \cup [s^{-n}(2 - \delta + \delta^2)/(2 - \delta), 1] \right)$, which has positive capacity for large n . Therefore we can assume that $\text{cap}(E^{**}) > 0$. The left-hand side of (6.2) is increasing if we replace $\mathbb{D} \setminus E^*$ by $(\mathbb{D} \setminus E^{**})$. The function $\omega(z, \mathbb{T}, \mathbb{D} \setminus E^{**})$ is harmonic in $\mathbb{D} \setminus E^{**}$ and equals 1 on $\mathbb{T}$, therefore it is comparable with $g_{\overline{\mathbb{C}} \setminus E^{**}}$. More precisely, let $A = \inf_{\mathbb{T}} g_{\overline{\mathbb{C}} \setminus E^{**}}$. It follows from Harnack's principle that there is a constant c_δ depending only on δ such that $g_{\overline{\mathbb{C}} \setminus E^{**}}(z) \leq c_\delta A$ for $z \in \mathbb{T}$. Thus

$$(6.3) \quad A\omega(z, \mathbb{T}, \mathbb{D} \setminus E^{**}) \leq g_{\overline{\mathbb{C}} \setminus E^{**}}(z) \leq c_\delta A\omega(z, \mathbb{T}, \mathbb{D} \setminus E^{**})$$

holds true on $\mathbb{T}$ and an application of the maximum principle shows that it is true for all $z \in \mathbb{D} \setminus E^{**}$. This implies

$$(6.4) \quad \omega(w, \mathbb{T}, \mathbb{D} \setminus E^*) \leq \frac{1}{A} g_{\overline{\mathbb{C}} \setminus E^{**}}(w)$$

for $w \in \mathbb{T}_{d_6 q^{n-m}}$. It follows from the definition of the Green function (2.1) that if $F \subset \mathbb{R}$ and $z = x + iy$, $x, y \in \mathbb{R}, y > 0$ then $g_{\overline{\mathbb{C}} \setminus F}(z)$ is monotone increasing in y . Using also the symmetry with respect to the real axis we have

$$(6.5) \quad \sup_{w \in \mathbb{T}_{d_6 q^{n-m}}} g_{\overline{\mathbb{C}} \setminus E^{**}}(w) \leq \sup_{w \in L} g_{\overline{\mathbb{C}} \setminus E^{**}}(w),$$

where $L = \{z : z = x + id_6 q^{n-m}, |x| \leq d_6 q^{n-m}\}$. By Harnack's inequality (2.5) and (6.3) we obtain

$$(6.6) \quad \sup_{w \in L} g_{\overline{\mathbb{C}} \setminus E^{**}}(w) \leq d_8 g_{\overline{\mathbb{C}} \setminus E^{**}}(id_6 q^{n-m}) \leq d_8 c_\delta A\omega(id_6 q^{n-m}, \mathbb{T}, \mathbb{D} \setminus E^{**}).$$

On the other hand, by Lemma 4 the right-hand side of (6.2) is not less than $d_9 q^{-m} \omega(0, \mathbb{T}, \mathbb{D} \setminus E^{**})$. In view of this and (6.3)–(6.6) it suffices to prove

$$(6.7) \quad \omega(id_6 q^{n-m}, \mathbb{T}, \mathbb{D} \setminus E^{**}) \leq d_{10} q^{-m} \omega(0, \mathbb{T}, \mathbb{D} \setminus E^{**}).$$

Now set $v(w) = (w + 1/w)/2$ and

$$u(v) = q^{n+2}v - \sqrt{(q^{n+2}v)^2 - 1}.$$

Under the mapping $z = u(v(w))$ the origin is mapped into itself, $\mathbb{D} \setminus E^{**}$ is mapped into a set containing $\mathbb{D}$, E^{**} is mapped into a subset of the unit circle $\mathbb{T}$ and the point $id_6 q^{n-m}$ is mapped into a point of distance $\geq dq^m$ from $\mathbb{T}$, where d is a constant. Thus with $z = u(v(w)) = h(w)$, using Harnack's inequality we get (6.7). This completes the proof of Theorem 5.

The proof of Corollary 1 is immediate from Lemmas 1 and 3. First of all, Lemma 1 implies (1.2) and

$$\lim_{r \rightarrow 0} \frac{\text{cap}(E \cap [-r, 0])}{r} = \frac{1}{4}.$$

Then, taking $I = [-r, 0]$, $J = [0, r]$ and $F = E \cap [-r, 0]$ in (7.10) we get

$$(6.8) \quad \lim_{r \rightarrow 0} \frac{\text{cap}((E \cap [-r, 0]) \cup [0, r])}{r} = \frac{1}{2}.$$

Next, taking $I = [0, r]$, $J = [-r, 0]$, $F = E \cap [0, r]$ and $G = E \cap [-r, 0]$ in (7.9) we can infer

$$(6.9) \quad 1 \geq \frac{\text{cap}(E \cap [-r, r])}{\text{cap}((E \cap [-r, 0]) \cup [0, r])} \geq \frac{4\text{cap}(E \cap [0, r])}{r} \rightarrow 1.$$

Finally, (1.10) is a direct consequence of (6.8) and (6.9). ■

7. Lemmas

Lemma 1. (1.5) for every $\varepsilon > 0$ implies (1.2).

Proof. Let $\eta > 0$ be arbitrary such that $1 + \eta \leq (1 - \varepsilon/2)/(1 - \varepsilon)$. For $t/(1 + \eta) \leq u \leq t$ we have $E_{\varepsilon/2}(u) \subseteq E_\varepsilon(t)$, therefore

$$\begin{aligned} \frac{1}{4} - \frac{\text{cap}(E_{\varepsilon/2}(u))}{u} &\geq \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{u} \geq \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} (1 + \eta) \\ &\geq \frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} - \eta. \end{aligned}$$

On adding η , dividing by u both sides and integrating with respect to u over the interval $t/(1 + \eta) \leq u \leq t$ we obtain

$$\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \leq \frac{1}{\log(1 + \eta)} \int_{t/(1+\eta)}^t \left(\frac{1}{4} - \frac{\text{cap}(E_{\varepsilon/2}(u))}{u} \right) \frac{1}{u} du + \eta.$$

Therefore, the finiteness of the integral in (1.5) (for $\varepsilon/2$ rather than for ε) gives

$$\limsup_{t \rightarrow 0} \left(\frac{1}{4} - \frac{\text{cap}(E_\varepsilon(t))}{t} \right) \leq \eta,$$

and since here $\eta > 0$ can be arbitrarily small, it follows that

$$(7.1) \quad \lim_{t \rightarrow 0} \frac{\text{cap}(E_\varepsilon(t))}{t} = \frac{1}{4}.$$

Now let $\{t_n\}$ be an arbitrary positive sequence tending to 0 and set

$$F_n = E_\varepsilon(t_n)/t_n, \quad \nu_n = \frac{1}{\mu_{F_n}([\varepsilon, 1 - \varepsilon])} \mu_{F_n} \Big|_{[\varepsilon, 1 - \varepsilon]}.$$

We have just proved that $\text{cap}(F_n) \rightarrow 1/4$ as $n \rightarrow \infty$, and below we verify that this implies the convergence $\mu_{F_n} \rightarrow \mu_{[0,1]}$ in the weak* topology. Since

$$\mu_{[0,1]}((\varepsilon, 1 - \varepsilon)) > 1 - 2\sqrt{\varepsilon},$$

there is an n_0 such that for $n \geq n_0$ we have $\mu_{F_n}((\varepsilon, 1 - \varepsilon)) \geq 1 - 2\sqrt{\varepsilon}$. There is also an n_1 such that for $n \geq n_1$ the inequality

$$U^{\mu_{F_n}}(x) = \log \frac{1}{\text{cap}(F_n)} \leq (1 + \varepsilon) \log 4, \quad x \in F_n$$

holds, which implies for $n \geq \max(n_0, n_1)$

$$U^{\nu_n}(x) \leq \frac{1}{1 - 2\sqrt{\varepsilon}} (1 + \varepsilon) \log 4, \quad x \in F_n.$$

But the measure ν_n is supported on $F_n \cap [\varepsilon, 1 - \varepsilon]$ and has mass 1, hence the preceding inequality gives

$$\log \frac{1}{\text{cap}(F_n \cap ([\varepsilon, 1 - \varepsilon]))} \leq \int U^{\nu_n} d\nu_n \leq \frac{1}{1 - 2\sqrt{\varepsilon}} (1 + \varepsilon) \log 4, \quad x \in F_n,$$

i.e.

$$\text{cap}((E_n \cap [0, t_n])/t_n) \geq \text{cap}(F_n \cap [\varepsilon, 1 - \varepsilon]) \geq \left(\frac{1}{4}\right)^{(1+\varepsilon)/(1-2\sqrt{\varepsilon})}.$$

Since here $\varepsilon > 0$ is arbitrary, it follows that $\text{cap}(E_n \cap [0, t_n])/t_n \rightarrow 1/4$ as $n \rightarrow \infty$, and this is (1.2).

Above we used that as $n \rightarrow \infty$, we have $\mu_{F_n} \rightarrow \mu_{[0,1]}$ in the weak* topology on measures. In fact, let σ be a weak* limit of some subsequence, say $\mu_{F_{n_l}} \rightarrow \sigma$ as $l \rightarrow \infty$. Then σ is supported in $[0, 1]$, has total mass 1, and all we have to show is that $\sigma = \mu_{[0,1]}$. We know that

$$(7.2) \quad U^{\mu_{F_n}}(x) = \log \frac{1}{\text{cap}(F_n)}$$

for $x \in F_n$ with the exception of a set of capacity 0, and the same is true for $[0, 1]$. Since $F_n \subset [0, 1]$, it follows that

$$U^{\mu_{F_n}}(x) \leq U^{\mu_{[0,1]}}(x) + \log \frac{\text{cap}([0, 1])}{\text{cap}(F_n)}$$

for $x \in F_n$ with the exception of a set of capacity 0, and since every set of zero capacity has zero μ_{F_n} -measure (see [6, Remark I.1.7, p. 28]), it follows that this inequality is true μ_{F_n} -almost everywhere. But then by the principle of domination [6, Theorem II.3.2] the same inequality is true for all $x \in \mathbb{C}$. Fixing such an $x \notin [0, 1]$ and letting n tend to infinity through the subsequence $\{n_l\}$ it follows from $\text{cap}(F_n) \rightarrow 1/4 = \text{cap}(E)$ that

$$U^\sigma(x) \leq U^{\mu_{[0,1]}}(x).$$

Thus, this inequality is true for all $x \in \mathbb{C} \setminus [0, 1]$.

However, the function

$$U^{\mu_{[0,1]}}(x) - U^\sigma(x)$$

vanishes at infinity, so it is harmonic there, and an appeal to the minimum principle on the domain $\overline{\mathbb{C}} \setminus [0, 1]$ yields that we must have

$$U^\sigma(x) \equiv U^{\mu_{[0,1]}}(x), \quad x \in \mathbb{C} \setminus [0, 1].$$

Now we can conclude $\sigma = \mu_{[0,1]}$ from the unicity theorem [6, Theorem II.4.13]. ■

Lemma 2. *Let $J = \{e^{i\varphi} : \pi/3 \leq \varphi \leq 2\pi/3\}$ be the middle third of the upper part of the unit circle. For every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ with the following property: if $F \subset [-1, 1]$ is any compact set with $[-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1] \subseteq F$, then for $x \in [-1, 1] \setminus F$ the inequality*

$$(7.3) \quad \omega(x, J, \mathbb{D} \setminus F) \geq c_\varepsilon \omega(x, \mathbb{T}, \mathbb{D} \setminus F)$$

holds.

Remark 1. The proof actually works the same way for any arc $J \subset \mathbb{T}$.

Proof. First we verify the lemma in the special case when $[-1, 1] \setminus F = I = (u, v)$ is an interval. Let $\alpha \in I$ be the point for which

$$-\frac{u - \alpha}{1 - \alpha u} = \frac{v - \alpha}{1 - \alpha v},$$

and apply the conformal map $\psi_1(z) = (z - \alpha)/(1 - \alpha z)$. This maps the unit circle into itself, F into a set F' of type $[-1, -a] \cup [a, 1]$, and J into some arc J' of the upper half circle $\mathbb{T}_+ = \{e^{i\varphi} : 0 \leq \varphi \leq \pi\}$ (see Figure 3). It is easy to see that there is a constant $b_\varepsilon > 0$ depending only ε such that F' contains the intervals $[-1, -1 + b_\varepsilon]$ and $[1 - b_\varepsilon, 1]$ and both the arc length of J' and the distance of J' from the points ± 1 is $\geq b_\varepsilon$. Map now $\mathbb{D} \setminus F'$ conformally onto $\mathbb{D}$ via the mapping ψ_2 normalized by $\psi_2(0) = 0, \psi_2'(0) > 0$. The image of $[-1, 0] \cap F' = [-1, -a]$ is an arc on $\mathbb{T}$ symmetric about the point -1 , and similarly the image of $[0, 1] \cap F' = [a, 1]$ is an arc on $\mathbb{T}$ symmetric about the point 1 , furthermore the length of these arcs are bounded from below by some constant $d_\varepsilon > 0$. $\mathbb{T}$ is mapped into the complementary arcs of $\mathbb{T}$, and let us denote the complementary arc lying on the upper half plane by A'' (which is the image of the upper half circle $\mathbb{T}_+$ under ψ_2 , i.e. $A'' = \psi_2(\mathbb{T}_+)$). The image J'' of J' is a subarc of A'' , and its length is comparable to the length of the latter, i.e. with some $\delta_\varepsilon > 0$ we have

$$(\text{arc length of } J'') \geq \delta_\varepsilon (\text{arc length of } A'').$$

If $y = \psi_2(\psi_1(x)) \in (-1, 1)$ is the image of x , then using the conformal invariance of harmonic measures, (7.3) takes the form

$$\omega(y, J'', \mathbb{D}) \geq c_\varepsilon \omega(y, \psi_2 \circ \psi_1(\mathbb{T}), \mathbb{D}),$$

which, using the symmetry of the image $\psi_2 \circ \psi_1(\mathbb{T}) = A'' \cup (-A'')$, is equivalent to

$$\omega(y, J'', \mathbb{D}) \geq 2c_\varepsilon \omega(y, A'', \mathbb{D}).$$

But in $\mathbb{D}$ harmonic measures are given by the Poisson kernel, so the preceding inequality is the same as

$$(7.4) \quad \frac{1}{2\pi} \int_{J''} \frac{1-y^2}{|\xi-y|^2} |d\xi| \geq 2c_\varepsilon \frac{1}{2\pi} \int_{A''} \frac{1-y^2}{|\xi-y|^2} |d\xi|,$$

which is clear with some $c_\varepsilon > 0$, since $y \in [-1, 1]$ and on the two sides during integration ξ runs through two arcs of comparable length both of which lie at distance $\geq d_\varepsilon/4$ from $[-1, 1]$. Thus, (7.4) is true with some $c_\varepsilon > 0$, and this gives (7.3).

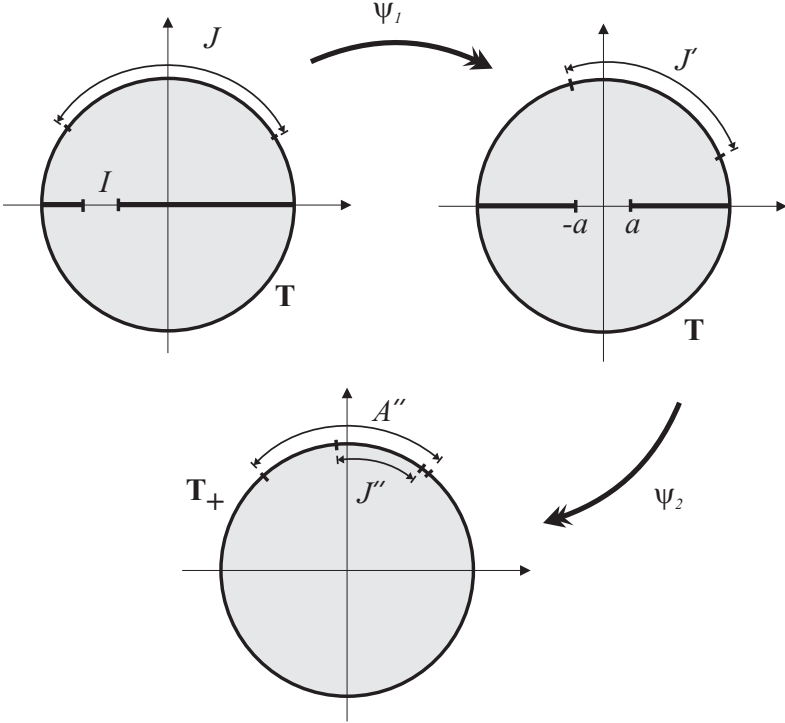


Figure 3: the mappings ψ_1 and ψ_2

Next we turn to the general case, i.e. when $[-1, 1] \setminus F = [-1 + \varepsilon, 1 - \varepsilon] \setminus F$ is an arbitrary open set. Since the constant c_ε should be independent of the set F (depending only on ε with $[-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1] \subseteq F$), without loss of generality we may assume F to consist of finitely many intervals, in which case $[-1, 1] \setminus F$ consists of finitely many open intervals, say $I_1, \dots, I_m$.

According to (2.4), what we have to prove is that there is a constant $c_\varepsilon > 0$ such that for $x \in [-1, 1] \setminus F$ we have

$$(7.5) \quad \text{Bal}(\delta_x, \mathbb{D} \setminus F)(J) \geq c_\varepsilon \text{Bal}(\delta_x, \mathbb{D} \setminus F)(\mathbb{T}).$$

We show that the constant c_ε verified above for the special case when $[-1, 1] \setminus F$ was an interval, is appropriate. To this end, starting from $\nu_0 = \delta_x$, we successively define the measures ν_n by

$$\nu_{n+1} = \text{Bal}(\nu_n, \mathbb{D} \setminus ([-1, 1] \setminus I_{j_n})),$$

where $j_n \in \{1, 2, \dots, m\}$ is the index j for which $\nu_n(I_j)$ is maximal for $j = 1, \dots, m$. Each ν_n is supported on $\mathbb{T} \cup [-1, 1]$, and on $\mathbb{T} \cup F$ the measures ν_n form a monotone increasing sequence of measures. Note also that on $\mathbb{T} \cup F$ we have

$$\nu_{n+1} - \nu_n = \int_{I_{j_n}} \text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I_{j_n})) d\nu_n(y),$$

and by the special case proved in the first part of this proof, here we have for all $x \in I_{j_n}$ the inequality

$$\text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I_{j_n}))(J) \geq c_\varepsilon \text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I_{j_n}))(\mathbb{T}).$$

Therefore the same is true of $\nu_{n+1} - \nu_n$, i.e. we have

$$\nu_{n+1}(J) - \nu_n(J) \geq c_\varepsilon(\nu_{n+1}(\mathbb{T}) - \nu_n(\mathbb{T})), \quad n = 0, 1, \dots$$

Since $\nu_0(J) = \nu_0(\mathbb{T}) = 0$, induction gives

$$\nu_{n+1}(J) \geq c_\varepsilon \nu_{n+1}(\mathbb{T}), \quad n = 0, 1, \dots,$$

therefore (7.5) will follow from here if we show that $\nu_n \rightarrow \text{Bal}(\delta_x, \mathbb{D} \setminus F)$ as $n \rightarrow \infty$. As $\{\nu_n|_{\mathbb{T} \cup F}\}_{n=0}^\infty$ is an increasing sequence of measures on $\mathbb{T} \cup F$, it converges to some measure ν supported on $\mathbb{T} \cup F$, and to complete the proof we show that $\nu = \text{Bal}(\delta_x, \mathbb{D} \setminus F)$ and $\nu_n([-1, 1] \setminus F) \rightarrow 0$ as $n \rightarrow \infty$. Since the total mass of each ν_n is 1, it is clear that the total mass of ν is at most 1. Also, by the properties of balayage measures, for $z \in C \cup F$ and for all n we have the equality

$$U^{\nu_{n+1}}(z) = U^{\nu_n}(z) = \dots = U^{\nu_0}(z) = \log \frac{1}{|z - x|},$$

and it is easy to see that then the same is true of ν , i.e.

$$(7.6) \quad U^\nu(z) = \log \frac{1}{|z - x|}, \quad z \in \mathbb{T} \cup F.$$

Now $\text{Bal}(\delta_x, \mathbb{D} \setminus F)$ is the unique measure supported on $\mathbb{T} \cup F$ which has mass 1 and its logarithmic potential is $\log 1/|z - x|$, thus the proof will be complete if we show that ν has mass 1, i.e. $\nu(\mathbb{T} \cup F) = 1$, which is the same as

$$\lim_{n \rightarrow \infty} \nu_n([-1, 1] \setminus F) = 0$$

which we wanted to prove anyway. This will be done by showing that in each step when going from ν_n to ν_{n+1} a fixed portion of the mass $\nu_n|_{[-1, 1] \setminus F}$ is moved to F , i.e. with some $\gamma < 1$ we have

$$(7.7) \quad \nu_{n+1}([-1, 1] \setminus F) \leq \gamma \nu_n([-1, 1] \setminus F).$$

Let $I_j = [a_j, b_j]$, and let $\tau > 0$ be so small that all the intervals $[a_j - \tau, a_j]$ and $[b_j, b_j + \tau]$ are part of $(-1, 1)$ and they are disjoint. For $I = I_{j_n}$ and $y \in I$ the value

$$\text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I)) ([a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]),$$

which is the same as

$$\omega\left(y, [a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau], \mathbb{D} \setminus ([-1, 1] \setminus I)\right)$$

is bounded from below by a constant ρ independent of n and $y \in I = I_{j_n}$. In fact, consider the conformal maps ψ_1, ψ_2 from the first part of the proof. Under $\psi_2 \circ \psi_1$ the set $[a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]$ is mapped into the union of two arcs $A_{\pm}$, one-one around ± 1 , of length bounded from below by a positive constant depending only on ε and τ . Now the inequality

$$(7.8) \quad \omega(y, [a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau], \mathbb{D} \setminus ([-1, 1] \cup I)) \geq \rho$$

with some positive constant ρ follows from the fact that here the left-hand side is

$$\omega(z, A_- \cup A_+, \mathbb{D}) = \frac{1}{2\pi} \int_{A_- \cup A_+} \frac{1 - z^2}{|\xi - z|^2} |d\xi|, \quad z = \psi_2(\psi_1(y)),$$

and the integral is bounded from below by a positive constant ρ for any point $z \in [-1, 1]$ (and hence in particular also for the point $z = \psi_2(\psi_1(y))$).

We obtain from (7.8)

$$\begin{aligned} & \text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I)) (F) \\ & \geq \text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I)) ([a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]) \geq \rho, \end{aligned}$$

which gives

$$\begin{aligned} \text{Bal}\left(\nu_n|_{I_{j_n}}, \mathbb{D} \setminus ([-1, 1] \setminus I)\right)(F) &= \int_{I_{j_n}} \text{Bal}(\delta_y, \mathbb{D} \setminus ([-1, 1] \setminus I))(F) d\nu_n(y) \\ &\geq \int_{I_{j_n}} \rho d\nu_n(y) = \rho \nu_n(I_{j_n}), \end{aligned}$$

and here the right-hand side is at least $\rho \nu_n([-1, 1] \setminus F)/m$ by the choice of the interval I_{j_n} . Thus,

$$\begin{aligned} \nu_{n+1}([-1, 1] \setminus F) &\leq \nu_n([-1, 1] \setminus F) - \text{Bal}\left(\nu_n|_{I_{j_n}}, \mathbb{D} \setminus ([-1, 1] \setminus I)\right)(F) \\ &\leq \nu_n([-1, 1] \setminus F) - \rho \nu_n([-1, 1] \setminus F)/m = \nu_n([-1, 1] \setminus F)(1 - \rho/m). \end{aligned}$$

This proves (7.7) and the proof is complete. ■

Lemma 3. *Let I be a closed interval in $\mathbb{R}$ and let J be a closed interval that is attached either from the left or from the right to I . Let F and G be closed subsets of I and J , respectively. Then*

$$(7.9) \quad \frac{\text{cap}(F)}{|I|} \leq \frac{\text{cap}(F \cup G)}{4\text{cap}(I \cup G)}.$$

In particular, if $G = J$ then

$$(7.10) \quad \frac{\text{cap}(F)}{|I|} \leq \frac{\text{cap}(F \cup J)}{|I| + |J|}.$$

Proof. Without loss of generality we may assume F and G to be regular compact sets (or to consist of finitely many closed intervals if we wish). The equilibrium measure μ_F is obtained from μ_I by adding to $\mu_I|_F$ the balayage of $\nu := \mu_I|_{I \setminus F}$ out of $\mathbb{C} \setminus F$ (see [6, Theorem IV.1.6, (e)]), and in this balayage process the potential on F increases by a constant value. More precisely (see (2.2), (2.3)), for $x \in F$ and $\bar{\nu} := \text{Bal}(\nu, \mathbb{C} \setminus F)$ we have

$$U^{\bar{\nu}}(x) = U^\nu(x) + \int_{I \setminus F} g_{\mathbb{C} \setminus F}(a) d\nu(a),$$

and this gives

$$U^{\mu_F}(x) = U^{\mu_I}(x) + \int_{I \setminus F} g_{\mathbb{C} \setminus F}(a) d\mu_I(a).$$

Taking into account that for $x \in F$ the equilibrium potentials on the left and right-hand sides are the constants $\log 1/\text{cap}(F)$ and $\log 1/\text{cap}(I) = \log 4/|I|$, respectively, we obtain the identity

$$(7.11) \quad \log \frac{1}{\text{cap}(F)} - \log \frac{4}{|I|} = \int_{I \setminus F} g_{\overline{\mathbb{C}} \setminus F}(a) d\mu_I(a).$$

The analogous formula for $F \cup G$ and $I \cup G$ reads as

$$(7.12) \quad \begin{aligned} \log \frac{1}{\text{cap}(F \cup G)} - \log \frac{1}{\text{cap}(I \cup G)} &= \int_{(I \cup G) \setminus (F \cup G)} g_{\overline{\mathbb{C}} \setminus (F \cup G)}(a) d\mu_{I \cup G}(a) \\ &= \int_{I \setminus F} g_{\overline{\mathbb{C}} \setminus (F \cup G)}(a) d\mu_{I \cup G}(a), \end{aligned}$$

where we used that $(I \cup G) \setminus (F \cup G) = I \setminus F$, so the integration is over the same set on the right-hand sides of (7.11) and (7.12). Since the measure μ_I is the balayage of $\mu_{I \cup G}$ onto I (see [6, Theorem IV.1.6, (e)]), we have on $I \setminus F$ the inequality $d\mu_{I \cup G}(a) \leq d\mu_I(a)$. At the same time $g_{\overline{\mathbb{C}} \setminus (F \cup G)}(a) \leq g_{\overline{\mathbb{C}} \setminus F}(a)$, and these show that the integral on the right-hand side of (7.12) is not larger than the integral on the right-hand side of (7.11). This gives

$$\log \frac{1}{\text{cap}(F \cup G)} - \log \frac{1}{\text{cap}(I \cup G)} \leq \log \frac{1}{\text{cap}(F)} - \log \frac{4}{|I|},$$

which is the same as (7.9). ■

Lemma 4. *There is an ε_0 such that for each $0 < \varepsilon < \varepsilon_0$ there exists a constant C_ε with the following property: if $F \subset [-1, 1]$ then*

$$(7.13) \quad \omega(0, \mathbb{T}, \mathbb{D} \setminus F) \geq C_\varepsilon \omega(0, \mathbb{T}, (\mathbb{D} \setminus F) \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]).$$

Proof. Clearly, for every $\varepsilon > 0$

$$(7.14) \quad \begin{aligned} &\omega(0, \mathbb{T}, (\mathbb{D} \setminus F) \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]) \\ &\leq \omega(0, \mathbb{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1], (\mathbb{D} \setminus F) \setminus ([-1, -1 - \varepsilon] \cup [1 - \varepsilon, 1])). \end{aligned}$$

There is a conformal mapping Φ of $\mathbb{D} \setminus ([-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1])$ onto $\mathbb{D}$ which maps $\mathbb{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]$ onto $\mathbb{T}$, $[1 - \varepsilon, 1]$ into an arc of the form $L = \{e^{i\phi} : |\phi| \leq \delta_\varepsilon\}$ (and symmetrically $[-1, -1 + \varepsilon]$ into an opposite arc L'), it takes 0 into 0 and

$F \cap [-1 + \varepsilon, 1 - \varepsilon]$ into some subset $F^* \subset [-1, 1]$. Because of the conformal invariance of harmonic measures,

(7.15)

$$\omega(0, \mathbb{T} \cup [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1], (\mathbb{D} \setminus F) \setminus ([-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1])) = \omega(0, \mathbb{T}, \mathbb{D} \setminus F^*).$$

If ε is sufficiently small, then $\mathbb{T} \setminus (L \cup L')$ contains the pair of arcs $J = \{e^{i\phi} : \pi/4 \leq |\phi| \leq 3\pi/4\}$. According to [4, Lemma 7.2]

$$(7.16) \quad \frac{1}{2}\omega(0, \mathbb{T}, \mathbb{D} \setminus F^*) \leq \omega(0, J, \mathbb{D} \setminus F^*),$$

and here the right-hand side is at most $\omega(0, \mathbb{T} \setminus (L \cup L'), \mathbb{D} \setminus F^*)$. Now application of Φ^{-1} gives

$$\begin{aligned} \omega(0, \mathbb{T} \setminus (L \cup L'), \mathbb{D} \setminus F^*) &= \omega(0, \mathbb{T}, (\mathbb{D} \setminus F) \setminus ([-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1])) \\ &\leq \omega(0, \mathbb{T}, \mathbb{D} \setminus F). \end{aligned}$$

This, along with (7.14)–(7.16) gives (7.13) and the proof is complete. ■

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