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A topology for semiclosed operators in a Hilbert space

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Abstract. We introduce a topology in the set of all semiclosed operators in a Hilbert space and investigate the topological structure by using the method of quotients of bounded operators. Under our topology, it is shown that the set of all closed operators is open in the set of all semiclosed operators, and that the topology restricted to the set of all closed operators is strictly stronger than the topology induced from the gap metric.

1. Introduction

Let $(H, (\cdot,\cdot))$ be an infinite dimensional complex Hilbert space. We denote by $\mathcal{B}(H)$ the set of all bounded linear operators defined on H. A subspace $M \subseteq H$ is called *semiclosed* if there exists a Hilbert space norm $\|\cdot\|_M$ on M such that the inclusion mapping $(M, \|\cdot\|_M) \hookrightarrow H$ is continuous, in short, M is continuously embedded in H. It is well known (cf. [10, Lemma 1]) that M is semiclosed if and only if M is an operator range, namely there exists an operator $B \in \mathcal{B}(H)$ such that $M = BH = \{Bu : u \in H\}$.

A linear operator s (with its domain $\mathrm{dom}(s)$) in H is called *semiclosed* (resp. closed) if its graph $G(s) = \{(u, su) : u \in \mathrm{dom}(s)\}$ is a semiclosed (resp. closed) subspace in the product Hilbert space $H \times H$.

Kaufman [10] showed that the set of all semiclosed operators is the smallest family containing all closed operators and itself closed under the algebraic operations of addition and multiplication. Moreover, he showed that an operator s is semiclosed if and only if s is represented by a quotient B/A of bounded operators

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$A, B \in \mathcal{B}(H)$ with the kernel condition $\ker A \subseteq \ker B$, where B/A is defined by the mapping $Au \to Bu$, $u \in H$ (cf. [6]). A merit for the quotient representation of a semiclosed operator is to make use of the theory of bounded operators. Douglas's majorization theorem [4] is fundamental for our discussions.

In 1973, Caradus [3] introduced a locally convex Hausdorff topology in the set of all semiclosed operators in a Banach space, and he studied continuity of the addition and the product of those operators.

Our purpose of this paper is to introduce a locally convex Hausdorff topology in the set of all semiclosed operators in a Hilbert space. Using the quotient representation of a semiclosed operator, we study the topological properties of the set of those operators. We prove that the addition and the scalar multiplication in the set are continuous, and that the multiplication from the left side is continuous. We characterize all connected components of the set of semiclosed operators. For the set of all closed operators we show that the set is open in the set of all semiclosed operators. Our topology in the set of all closed operators is shown to be strictly stronger than that induced from the gap metric introduced in [8], [11]. It is also shown that if an operator s is closed then for any semiclosed operator t with $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$, $s(\kappa) = s + \kappa t$ is closed for every complex number κ with the absolute value sufficiently small.

2. Preliminaries

Throughout this paper, we denote by H an infinite dimentional Hilbert space and by $\mathcal{B}(H)$ the set of all bounded (linear) operators defined on H. For any $A \in \mathcal{B}(H)$, the de Branges space $(AH, (\cdot,\cdot)_A)$ induced from A is defined as the operator range AH equipped with a Hilbert space structure $(Au, Av)_A = (Pu, Pv)$ for $u, v \in H$, where P is the orthogonal projection onto the orthogonal complement $(\ker A)^{\perp}$ of $\ker A$. The following theorem shows a relation between the de Brange space and a semiclosed subspace, and is a key for our topology.

Theorem 2.1. ([1]). *Let M be a semiclosed subspace in H. Then, for any Hilbert space norm $\|\cdot\|_M$ such that M is continuously embedded in H, there is a unique positive bounded operator A such that $(M, \|\cdot\|_M)$ is isometrically isomorphic to the de Brange space $(AH, \|\cdot\|_A)$, (i.e. $M = AH$ and $\|x\|_M = \|x\|_A$ for $x \in M$). Conversely, for any positive bounded operator A, the operator range $M = AH$ with $\|x\|_M = \|x\|_A$ (for $x \in M$) is continuously embedded in H.*

Denote by $\mathcal{S}(H)$ the set of all semiclosed operators with their domains in H. Then the following theorem is a characterization of a semiclosed operator.

Theorem 2.2. ([10]). *Let s be an operator with* $\mathrm{dom}(s)$ *in H. Then the following conditions are equivalent.*

(i) *s is a member of $\mathcal{S}(H)$.*

(ii) *s is a quotient of bounded operators, namely there exist bounded operators $A, B \in \mathcal{B}(H)$ such that* $\ker A \subseteq \ker B$, $\mathrm{dom}(s) = AH$, $\mathrm{ran}(s) = BH$ *and*

$$s = B/A : Au \to Bu, \quad u \in H.$$

(iii) $\mathrm{dom}(s)$ *is a semiclosed subspace in H and the mapping*

$$s : (\mathrm{dom}(s), \|\cdot\|_s) \to H$$

is a bounded operator, where $\|\cdot\|_s$ is a Hilbert space norm on $\mathrm{dom}(s)$.

3. A topology for semiclosed operators

In this section, we introduce a metric in the set $\mathcal{S}(H)$ of all semiclosed operators. Since the domain of a semiclosed operator is a semiclosed subspace by Theorem 2.2 (iii), there are some Hilbert space norms on it. So let us choose and fix a norm from them for each semiclosed subspace. Let α be the correspondence between semiclosed subspaces and norms. Equivalently, we can consider α as the set of positive bounded operators each of which corresponds to a semiclosed subspace in the sense of Theorem 2.1.

Let s be a semiclosed operator with $\mathrm{dom}(s)$ in H. Then $\mathrm{dom}(s)$ has already been given a Hilbert space norm $\|\cdot\|_s = \|\cdot\|_{\alpha,s}$ belonging to α. Hence, by Theorem 2.1, there exists a unique positive bounded operator A such that $(\mathrm{dom}(s), \|\cdot\|_s)$ is isometrically isomorphic to $(AH, \|\cdot\|_A)$. The operator s is then uniquely represented up to α by a quotient B/A, so that we denote by $s \overset{\alpha}{=} B/A$.

Now we define an ε-neighborhood for $\varepsilon > 0$ of a semiclosed operator $s \overset{\alpha}{=} B/A$ by

$$V(s;\alpha,\varepsilon) = \{t \in \mathcal{S}(H) : t \overset{\alpha}{=} D/A,\ \|B - D\| < \varepsilon\}, \tag{3.1}$$

where $\|\cdot\|$ means an operator norm on $\mathcal{B}(H)$. Let $\mathcal{O}^\alpha$ be the topology induced from the neighborhood system as above. Note that if we take another β instead of α then the topologies $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$ in $\mathcal{S}(H)$ coincide (Remark 3.1). From now on we simply denote $\mathcal{O}^\alpha$ by $\mathcal{O}$.

For the topology $\mathcal{O}$ in $\mathcal{S}(H)$ we have the following fundamental property.

Theorem 3.1. *$\mathcal{S}(H) = (\mathcal{S}(H), \mathcal{O})$ is a locally convex Hausdorff space with the first countability.*

Proof. The first countability is clear by the definition (3.1) of $V(s;\alpha,\varepsilon)$. To show that $V(s;\alpha,\varepsilon)$ is convex, take $s_1, s_2 \in V(s;\alpha,\varepsilon)$ and let $s \overset{\alpha}{=} B/A$, $s_1 \overset{\alpha}{=} B_1/A$ and $s_2 \overset{\alpha}{=} B_2/A$. Since

$$ks_1 + (1-k)s_2 = kB_1/A + (1-k)B_2/A = \{kB_1 + (1-k)B_2\}/A$$

for $0 < k < 1$, we then have

$$\begin{aligned}\|B - \{kB_1 + (1-k)B_2\}\| &\le k\|B - B_1\| + (1-k)\|B - B_2\| \\ &< k\varepsilon + (1-k)\varepsilon = \varepsilon.\end{aligned}$$

Hence $ks_1 + (1-k)s_2 \in V(s;\alpha,\varepsilon)$.

To see the Hausdorff condition, we take any elements $s_1, s_2 \in \mathcal{S}(H)$ such that $s_1 \neq s_2$. Put $s_1 \overset{\alpha}{=} B_1/A_1$ and $s_2 \overset{\alpha}{=} B_2/A_2$. If $\mathrm{dom}(s_1) \neq \mathrm{dom}(s_2)$, then we have $V(s_1;\alpha,\varepsilon) \cap V(s_2;\alpha,\varepsilon) = \emptyset$ for any $\varepsilon > 0$, and if $\mathrm{dom}(s_1) = \mathrm{dom}(s_2)$, or $A_1 = A_2$, then $V(s_1;\alpha,\varepsilon) \cap V(s_2;\alpha,\varepsilon) = \emptyset$ for $\varepsilon = \|B_1 - B_2\|/2 > 0$. ■

Theorem 3.2. *Both addition and scalar multiplication are continuous in $\mathcal{S}(H)$.*

Proof. First put $s_1 \overset{\alpha}{=} B_1/A_1$ and $s_2 \overset{\alpha}{=} B_2/A_2$. Then by the definition of the sum of two quotients ([6, Theorem 3.1]) the domain $\mathrm{dom}(s_1 + s_2)$ is $A_1H \cap A_2H$, which is again an operator range AH (i.e., $AH = A_1H \cap A_2H$) for some positive operator A. So we assume $A \in \alpha$. Then we have

$$\begin{aligned}s_1 + s_2 &= B_1/A_1 + B_2/A_2 = B_1X_1/A_1X_1 + B_2X_2/A_2X_2 \\ &= B_1X_1/A + B_2X_2/A \overset{\alpha}{=} (B_1X_1 + B_2X_2)/A\end{aligned}$$

for some X_1, $X_2 \in \mathcal{B}(H)$ such that $A = A_1X_1$ and $A = A_2X_2$. For any $\varepsilon > 0$, put $\varepsilon_1 = \varepsilon/2\|X_1\|$ and $\varepsilon_2 = \varepsilon/2\|X_2\|$; then we have

$$V(s_1;\alpha,\varepsilon_1) + V(s_2;\alpha,\varepsilon_2) \subseteq V(s_1 + s_2;\alpha,\varepsilon).$$

In fact, taking any element $t_i \overset{\alpha}{=} D_i/A_i \in V(s_i;\alpha,\varepsilon_i)$ $(i = 1,2)$, we see

$$t_1 + t_2 = D_1/A_1 + D_2/A_2 \overset{\alpha}{=} (D_1X_1 + D_2X_2)/A$$

for the same X_1 and X_2 as above, so we have

$$\begin{aligned}\|(B_1X_1+B_2X_2)-(D_1X_1+D_2X_2)\| &\le \|B_1-D_1\|\|X_1\|+\|B_2-D_2\|\|X_2\|\\ &< \varepsilon_1\|X_1\|+\varepsilon_2\|X_2\|=\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

Hence addition is continuous. Next to show the continuity of scalar multiplication, let $k\in\mathbf{C}$ be a complex number, $\varepsilon>0$, $s\overset{\alpha}{=}B/A\in\mathcal{S}(H)$, and put

$$\varepsilon_1=\min\{1,\varepsilon/(1+\|B\|+|k|)\}. \tag{3.2}$$

Then we have

$$\{l\in\mathbf{C}:|k-l|<\varepsilon_1\}V(s;\alpha,\varepsilon_1)\subseteq V(ks;\alpha,\varepsilon).$$

In fact, for any $l'\in\{l\in\mathbf{C}:|k-l|<\varepsilon_1\}$ and $t\overset{\alpha}{=}D/A\in V(s;\alpha,\varepsilon_1)$, we see that $ks\overset{\alpha}{=}kB/A$, $l't\overset{\alpha}{=}lD/A$ and

$$\begin{aligned}\|l'D-kB\|=\|l'D-kD+kD-kB\| &\le |l'-k|\|D\|+|k|\|D-B\|\\ &<\varepsilon_1(\varepsilon_1+\|B\|)+|k|\varepsilon_1\le\varepsilon_1(1+\|B\|)+|k|\varepsilon_1 \qquad \text{by (3.2)}\\ &\le\varepsilon_1(1+\|B\|+|k|)\le\varepsilon.\end{aligned}$$

Hence, scalar multiplication is continuous.

■

Theorem 3.3. *If $t_n\to t$ in $\mathcal{S}(H)$, then $t_ns\to ts$ for any $s\in\mathcal{S}(H)$ as $n\to\infty$.*

Proof. Let $s\overset{\alpha}{=}B/A$, $t\overset{\alpha}{=}D/C$. Then by the definition of the product of two quotients ([6], Theorem 3.2), $ts=(D/C)(B/A)=DX_1/AY_1$ for some $X_1,Y_1\in\mathcal{B}(H)$ such that $Y_1H=B^{-1}(CH)$ and $CX_1=BY_1$. Let $\mathrm{dom}(ts)=EH$ for some $E\in\alpha$. Then since $AY_1H=EH$, there is a $Y_2\in\mathcal{B}(H)$ satisfying $AY_1Y_2=E$, so that by the definition of the quotient we see

$$DX_1/AY_1=DX_1Y_2/AY_1Y_2=DX_1Y_2/E.$$

Hence $ts=DX/E$ for $X=X_1Y_2$, or, more precisely, $ts\overset{\alpha}{=}DX/E$. Now for any neighborhood $V(ts;\alpha,\varepsilon)$ of ts, we have, putting $\delta=\varepsilon/\|X\|$,

$$V(t;\alpha,\delta)s\subseteq V(ts;\alpha,\varepsilon).$$

In fact, since $\mathrm{dom}(ts) = \mathrm{dom}(\tilde{t}s)$ for any $\tilde{t} \in V(t;\alpha,\delta)$ $(\tilde{t} \overset{\alpha}{=} \tilde{D}/C)$, we see that

$$\tilde{t}s = (\tilde{D}/C)(B/A) = \tilde{D}X_1/AY_1 = \tilde{D}X_1Y_2/AY_1Y_2 \overset{\alpha}{=} \tilde{D}X/E$$

for the same X_1, Y_1 and $Y_2 \in \mathcal{B}(H)$ as above. Hence

$$\|\tilde{D}X - DX\| \leq \|\tilde{D} - D\|\|X\| < \delta\|X\| = \varepsilon.$$

This implies $\tilde{t}s \in V(ts;\alpha,\varepsilon)$.

■

Let $s \overset{\alpha}{=} B/A \in \mathcal{S}(H)$. Then by Theorem 2.2 (iii) s is a bounded operator on $\mathrm{dom}(s)$, or equivalently, on AH with the de Brange norm $\|\cdot\|_A$ by Theorem 2.1, so we define an operator norm $\|s\|_\alpha$ of s by

$$\|s\|_\alpha = \sup_{u \in AH} \frac{\|su\|}{\|u\|_A}. \tag{3.3}$$

Lemma 3.4. *For $s \overset{\alpha}{=} B/A$, the following facts hold.*

$$\|s\|_\alpha = \|B\| \tag{3.4}$$

and

$$\begin{aligned} V(s;\alpha,\varepsilon) &= \{t \in \mathcal{S}(H) : t \overset{\alpha}{=} D/A,\ \|B - D\| < \varepsilon\} \\ &= \{t \in \mathcal{S}(H) : \mathrm{dom}(s) = \mathrm{dom}(t), \|s - t\|_\alpha < \varepsilon\}. \end{aligned} \tag{3.5}$$

Proof. For (3.4), note that $(\ker A)^\perp \supseteq (\ker B)^\perp$ from the kernel condition between A and B. Hence

$$\|s\|_\alpha = \sup_{v \in (\ker A)^\perp} \frac{\|Bv\|}{\|Av\|_A} = \sup_{v \in (\ker A)^\perp} \frac{\|Bv\|}{\|v\|} = \|B\|.$$

For (3.5), note that $\mathrm{dom}(s) = \mathrm{dom}(t) = AH$ if and only if t lies in a neighbourhood of s. Hence by (3.4) and the relation $(\ker A)^\perp \supseteq (\ker(B - D))^\perp$ we can see that $\|s - t\|_\alpha = \|B - D\|$, which implies (3.5).

■

Let M be a semiclosed subspace of H. Then by definition, M becomes a Hilbert space with respect to a norm belonging to α. We denote by $\mathcal{B}(M, H)$ the set of all bounded operators from such a Hilbert space M into H. Then $\mathcal{B}(M, H)$ is a Banach space with the operator norm defined by (3.3). Now let

$$\mathcal{S}_M = \{s \in \mathcal{S}(H) : \mathrm{dom}(s) = M\}.$$

Then $\mathcal{S}(H)$ is decomposed into the subsets $\mathcal{S}_M$ for all semiclosed subspaces M. For $\mathcal{S}_M$ we have:

Theorem 3.5. *Each $\mathcal{S}_M$ is a connected component of $\mathcal{S}(H)$, and is homeomorphic to $\mathcal{B}(M,H)$. In particular, $\mathcal{B}(H)$ is a connected component of $\mathcal{S}(H)$.*

Proof. To see that $\mathcal{S}_M$ is closed, and open too, first suppose that a sequence $s_n \in \mathcal{S}(H)$ $(n \geq 1)$ with $\operatorname{dom}(s_n) = M$ has a limit s in $\mathcal{S}(H)$. Then by the definition of a neighborhood of s, $\operatorname{dom}(s) = \operatorname{dom}(s_n)$ for all sufficiently large n, so that $\mathcal{S}_M$ is closed. Next take any $t \in \mathcal{S}(H)$ with $\operatorname{dom}(t) = M$. Then for any $\varepsilon > 0$, it is clear that

$$V(t;\alpha,\varepsilon) \subseteq \{s \in \mathcal{S}(H) : \operatorname{dom}(s) = M\} = \mathcal{S}_M.$$

Thus $\mathcal{S}_M$ is open. Moreover it follows from Lemma 3.4 that $\mathcal{S}_M$ is homeomorphic to the Banach space $\mathcal{B}(M,H)$ which is arcwise connected, so that $\mathcal{S}_M$ is a connected component.

■

Remark 3.1. Any two Hilbert space norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a semiclosed subspace M which make M complete are equivalent, that is, there exist constants $c > 0$ and $d > 0$ such that

$$c\|u\|_2 \leq \|u\|_1 \leq d\|u\|_2,\ u \in M.$$

The corresponding two norms of an operator in $\mathcal{B}(M,H)$ are also equivalent : For a semiclosed operator s with $\operatorname{dom}(s) = M$, since $\|s\|_i = \sup_{u\in M} \|su\|/\|u\|_i$ $(i = 1,2)$, we have

$$\frac{1}{d}\|s\|_2 \leq \|s\|_1 \leq \frac{1}{c}\|s\|_2, \quad s \in \mathcal{B}(M,H).$$

Thus, each component $\mathcal{B}(M,H)$ has a topology which is independent of any Hilbert space norm on M.

Corollary 3.6. *$\mathcal{S}(H)$ is metrizable.*

Proof. It follows from Theorem 3.5 that the real-valued function $d(\cdot,\cdot)$ depending on α defined below is a metric in $\mathcal{S}(H)$.

$$(3.6) \qquad d(s,t) = \begin{cases} 1 & (\operatorname{dom}(s) \neq \operatorname{dom}(t)) \\ \dfrac{\|s-t\|_\alpha}{1+\|s-t\|_\alpha} & (\operatorname{dom}(s) = \operatorname{dom}(t)) \end{cases}.$$

■

Theorem 3.7. *The set $\mathcal{C}(H)$ of all closed operators with their domains in H is open in $\mathcal{S}(H)$.*

Proof. By [9, Theorem 1], a quotient B/A is closed if and only if $A^*H + B^*H$ $(= (A^*A + B^*B)^{1/2}H)$ is a closed subspace in H. So let $s \overset{\alpha}{=} B/A \in \mathcal{C}(H)$; then $R_s = (A^2 + B^*B)^{1/2}$ has closed range, or equivalently,

$$\gamma_s := \inf\{\|R_s u\| : u \in (\ker R_s)^\perp, \|u\| = 1\} > 0. \tag{3.7}$$

Let $t \overset{\alpha}{=} D/A \in \mathcal{S}(H)$ be sufficiently close to s. Then we have to show that t is also closed, namely $R_t = (A^2 + D^*D)^{1/2}$ has closed range. First note that the kernels of R_t and R_s coincide. For, since $\ker A \subseteq \ker B$ and $\ker A \subseteq \ker D$ from the kernel conditions of the quotients B/A and D/A, we see that

$$\ker R_t = \ker R_s (= \ker A). \tag{3.8}$$

If D is close to B, then R_t is also close to R_s, so that for a sufficiently small $\varepsilon > 0$ we have $\|R_t - R_s\| < \varepsilon$ and $\gamma_s - \varepsilon > 0$. Hence we have

$$\varepsilon > \|R_t - R_s\| \geq \|R_t u - R_s u\| \geq \Big| \|R_t u\| - \|R_s u\| \Big|$$

for all unit vectors $u \in (\ker R_s)^\perp$. We then obtain

$$\|R_t u\| \geq \|R_s u\| - \varepsilon \geq \gamma_s - \varepsilon > 0.$$

Since $(\ker R_t)^\perp = (\ker R_s)^\perp$ by (3.8), we see from the above inequalities that R_t has closed range and $\gamma_t \geq \gamma_s - \varepsilon$. Hence we have

$$V(s; \alpha, \delta) = \{t \in \mathcal{S}(H) : t \overset{\alpha}{=} D/A,\ \|B - D\| < \delta\} \subseteq \mathcal{C}(H)$$

for a sufficiently small $\delta > 0$. Thus $\mathcal{C}(H)$ is open.

■

In [8], the gap metric has been introduced in $\mathcal{C}(H)$ and the distance between two closed operators s and t induced from the gap metric is defined as $\|P_s - P_t\|$, where P_s and P_t are the orthogonal projections onto the graphs $G(s)$ and $G(t)$ in $H \times H$, respectively. Now we show a relation between the metric d defined before and the gap metric.

Theorem 3.8. *In $\mathcal{C}(H)$, the topology induced from the metric d is strictly stronger than that induced from the gap metric.*

To prove this, we prepare a lemma.

Lemma 3.9. *If $s \overset{\alpha}{=} B/A$ is closed (hence $R_s = (A^2 + B^*B)^{1/2}$ has closed range), then the orthogonal projection P_s onto the graph $G(s)$ of s in $H \times H$ is given by*

$$P_s = \begin{pmatrix} A(R_s^\dagger)^2A & A(R_s^\dagger)^2B^* \\ B(R_s^\dagger)^2A & B(R_s^\dagger)^2B^* \end{pmatrix}, \tag{3.9}$$

where $R_s^\dagger$ is the generalized inverse of R_s ([2]).

Proof. By [7, Lemma 2.3], (since s is closed) we can write $s = B_l/A_l$ where $X = A_l$ and $Y = B_l$ are respectively the (unique) solutions of

$$A = XR_s \quad \text{and} \quad B = YR_s$$

with some kernel conditions. Let $V = \begin{pmatrix} A_l & 0 \\ B_l & 0 \end{pmatrix}$. Then the range $V(H \times H)$ of V is the graph $G(s)$ in $H \times H$. Note that

$$V^*V = \begin{pmatrix} A_l^*A_l + B_l^*B_l & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_{R_s} & 0 \\ 0 & 0 \end{pmatrix},$$

where $P_{R_s} = A_l^*A_l + B_l^*B_l$ is the orthogonal projection onto $(R_sH)^- = R_sH$ [7, Lemma 2.1 (1)]. Hence V^*V is a projection, so that V is a partial isometry. Hence

$$VV^* = \begin{pmatrix} A_lA_l^* & A_lB_l^* \\ B_lA_l^* & B_lB_l^* \end{pmatrix}$$

is also an orthogonal projection, and is one onto the graph $G(s)$ of s. Now since R_s has closed range, there is the generalized inverse $R_s^\dagger$ of R_s, and we have $A_l = AR_s^\dagger$ and $B_l = BR_s^\dagger$. Hence we obtain the desired operator matrix (3.9).

■

Proof of Theorem 3.8. We have to show that if closed operators $s_n \overset{\alpha}{=} B_n/A$ converge to a closed operator $s \overset{\alpha}{=} B/A$, that is, $B_n \to B$ as $n \to \infty$, then $P_{s_n} \to P_s$. Let $R_{s_n} = (A^2 + B_n^* B_n)^{1/2}$ and $R_s = (A^2 + B^* B)^{1/2}$. Here we may assume that $\ker R_{s_n} = \ker R_s$ by (3.8). Now, since $R_{s_n} \to R_s$, if we show the condition

$$\sup_n \|R_{s_n}^\dagger\| < \infty, \tag{3.10}$$

then we have $R_{s_n}^\dagger \to R_s^\dagger$ from [5, Proposition 2.3], and this assures the convergence $P_{s_n} \to P_s$. So, to see condition (3.10), let $\varepsilon > 0$ be small enough to have $\gamma_s - \varepsilon > 0$ (γ_s is the lower bound of $\|R_s u\|$ defined in (3.7)). Then there exists a natural number N such that

$$\varepsilon > \|R_s - R_{s_n}\| \geq \|R_s u - R_{s_n} u\| \geq \Big| \|R_s u\| - \|R_{s_n} u\| \Big|$$

for any $n \geq N$ and any unit vector $u \in (\ker R_{s_n})^\perp = (\ker R_s)^\perp$. Hence

$$\|R_{s_n} u\| \geq \|R_s u\| - \varepsilon \geq \gamma_s - \varepsilon > 0, \quad \text{or} \quad \gamma_{s_n} \geq \gamma_s - \varepsilon \quad \text{for} \quad n \geq N.$$

Therefore

$$(\gamma_s - \varepsilon)^{-1} \geq \gamma_{s_n}^{-1} = \|R_{s_n}^\dagger\|$$

(by [2, p. 325]), which implies (3.10).

■

Finally, in order to exclude the possibility that the metric d and the gap metric generate the same topology, we shall later show an example (Example 3.1) of a sequence which converges in the gap metric, but does not converge in the metric d.

Let t be a closed and densely defined operator in H. Then by Kaufman ([9, Corollary]) we can write $t = B/(I - B^* B)^{1/2}$ with a unique contraction B such that $\ker(I - B^* B) = 0$. In particular, t represents a bounded operator if and only if $\|B\| < 1$.

Related to this representation, in [11], K-convergence of such operators was defined as follows: $t_n \xrightarrow{K} t$ if $B_n \to B$ in $\mathcal{B}(H)$, where B_n and B are contractions corresponding to t_n and t respectively. As shown in [9] (or directly), the topology induced by K-convergence is strictly stronger than the topology induced from the gap metric. By using this fact, we have the following example.

Example 3.10. Let B be a bounded operator such that $\|B\| = 1$ and $\ker(I - B^*B) = \{0\}$. Then putting $B_n = ((n-1)/n)B$ for positive integers $n \geq 1$, we have

$$\|B_n^* B_n\| = \|B_n\|^2 = \left\| \frac{n-1}{n} B \right\|^2 = \left(\frac{n-1}{n} \right)^2 < 1,$$

so that B_n is a contraction with $\ker(I - B_n^* B_n) = \{0\}$. It is clear that $B_n \to B$ as $n \to \infty$, or $t_n \xrightarrow{K} t$, where t_n and t are bounded and closed (unbounded) operators corresponding to B_n and B respectively. This implies that $t_n \to t$ in the gap metric. On the other hand, since $\mathrm{dom}(t) \neq \mathrm{dom}(t_n) = H$ for any n, we see that t_n does not converge to t in the metric d.

4. Distance between semiclosed operators

Let s and t be any semiclosed operators such that $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$. Put $s \overset{\alpha}{=} B/A$ and $t \overset{\alpha}{=} D/C$. Then $AH = \mathrm{dom}(s) \subseteq \mathrm{dom}(t) = CH$. Hence, there exists a bounded operator X such that $A = CX$ by [4], so that

$$\begin{aligned} s + t &= B/A + D/C = B/A + DX/CX \\ &= B/A + DX/A \overset{\alpha}{=} (B + DX)/A, \end{aligned}$$

and

$$\begin{aligned} \|s - (s+t)\|_\alpha &= \sup_{u \in \mathrm{dom}(s) = AH} \frac{\|su - (s+t)u\|}{\|u\|_A} = \sup_{u = Av,\ v \in (\ker A)^\perp} \frac{\|su - (s+t)u\|}{\|u\|_A} \\ &= \sup_{v \in (\ker A)^\perp} \frac{\|Bv - (B+DX)v\|}{\|Av\|_A} = \sup_{v \in (\ker A)^\perp} \frac{\|Bv - (B+DX)v\|}{\|v\|} \\ &= \sup_{v \in (\ker A)^\perp} \frac{\|DXv\|}{\|v\|} = \|DX\| \quad (\text{by } (\ker DX)^\perp \subseteq (\ker A)^\perp). \end{aligned}$$

Write $t|_s = t|_{\mathrm{dom}(s)}$, the restriction of t to $\mathrm{dom}(s)$. Then we have $\|DX\| = \|t|_s\|_\alpha$. For, since $t|_s = DX/CX \overset{\alpha}{=} DX/A$ (by the same X as above), it follows $\|t|_s\|_\alpha = \|DX\|$ from (3.4). Now we have the following:

Theorem 4.1. *Let d be the metric (defined before) in $\mathcal{S}(H)$. Then, for any $s, t \in \mathcal{S}(H)$ such that* $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$, *we have*

$$d(s, s+t) = \frac{\|t|_s\|_\alpha}{1 + \|t|_s\|_\alpha}.$$

Corollary 4.2. *Suppose that* s *is closed and* t *is semiclosed with* $\operatorname{dom}(s) \subseteq \operatorname{dom}(t)$. *Then* $s(\kappa) = s + \kappa t$ *is closed for any complex number* κ *with its absolute value sufficiently small.*

Proof. Since
$$d(s, s(\kappa)) = \frac{|\kappa| \| t|_s \|_\alpha}{1 + |\kappa| \| t|_s \|_\alpha},$$
we see that for any sufficiently small $|\kappa|$, the distance $d(s, s(\kappa))$ is small enough for $s(k)$ to lie in a neighbourhood of s which is contained in $C(H)$ by Theorem 3.7. ■

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