*Acta Sci. Math.* (*Szeged*)
**79** (2013), 87–105

# An elementary method for the study of Meissner's equation and its application to proving the Oscillation Theorem

László Hatvani*

*Dedicated to the memory of Professor Béla Szőkefalvi-Nagy on his 100th birthday*

*Communicated by Á. Kurusa*

**Abstract.** A geometric method is presented to describe the dynamics of the linear second order differential equation with step function coefficient

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \text{ if } t_{k-1} \le t < t_k \qquad (k \in \mathbb{N}),$$

where $a_k > 0$, $t_0 = 0$, $t_k \nearrow \infty$ as $k \to \infty$. We rewrite this equation into a discrete dynamical system on the plane. The method is applied to the Meissner equation $x'' + \lambda^2 Q(t)x = 0$, where $\lambda > 0$ is a real parameter; $Q$ is a $2L$-periodic real function which is 1 on $[0, 2)$ and $a^2$ on $[2, 2L)$; $a$, $L$ ($0 < a \ne 1$, $L > 1$) are given constants. We give a complete elementary proof for the classical oscillation theorem on the $2L$-periodic and $4L$-periodic solutions of this equation not using even Floquet's theorem from the theory of differential equations.

Received March 4, 2013, and in revised form April 23, 2013.
*AMS Subject Classifications* (2000): 34B24, 34B30, 34C10; 74H45.
*Key words and phrases:* second order linear differential equations, non-autonomous equations, equations with periodic coefficients, Hill's equation, step function coefficients, eigenvalues of first and second type, oscillation.
* Supported by the Hungarian NFSR (OTKA K75517) and Analysis and Stochastics Research Group of the Hungarian Academy of Sciences, and by the European Union and co-funded by the European Social Fund under the project "Telemedicine-focused research activities on the field of Matematics, Informatics and Medical sciences" of project number "TÁMOP-4.2.2.A-11/1/KONV-2012-0073".

## 1. Introduction

The second order linear differential equation

$$x'' + P(t)x = 0,$$

where $P : [0, \infty) \to \mathbb{R}$ is a $T$-periodic function ($T > 0$), is known as Hill's equation. It arises in the mathematical physics [11] and often contains an eigenvalue real parameter $\lambda$:

$$x'' + \lambda P(t)x = 0. \tag{1.1}$$

The special values of $\lambda$ for which (1.1) has $T$-periodic (respectively, $2T$-periodic) solutions are called the *eigenvalues of the first kind* (respectively, *of the second kind*) of (1.1) [14]. The corresponding $T$-periodic (respectively, $2T$-periodic) solutions are called *eigenfunctions*. Here, and in what follows, by a $T$-periodic solution we mean a solution with the minimal positive period $T$.

The central theorem of the theory of Hill's equation is the so-called Oscillation Theorem, proved independently by A. M. Lyapunov [13] and O. Haupt [9], [10] (see also [3, Theorem 3.1]).

**Theorem A.** *If $P$ is piecewise continuous and positive, then the eigenvalues of the first kind $\{\lambda_i\}_{i=0}^{\infty}$ and the eigenvalues of the second kind $\{\widetilde{\lambda}_i\}_{i=1}^{\infty}$ form sequences such that*

$$-\infty < \lambda_0 < \widetilde{\lambda}_1 \le \widetilde{\lambda}_2 < \lambda_1 \le \lambda_2 < \widetilde{\lambda}_3 \le \widetilde{\lambda}_4 < \lambda_3 \le \lambda_4 < \cdots \tag{1.2}$$

*and* $\lim_{n\to\infty} \lambda_n = \infty$.

*For $\lambda = \lambda_0$ there exists a unique eigenfunction $\psi_0$. (Here, and in what follows, two eigenfunctions are distinguished only if they are linearly independent.) If $\lambda_{2i+1} < \lambda_{2i+2}$ for some $i \ge 0$, then there is a unique eigenfunction $\psi_{2i+1}$ at $\lambda = \lambda_{2i+1}$ and a unique eigenfunction $\psi_{2i+2}$ at $\lambda = \lambda_{2i+2}$. If, however, $\lambda_{2i+1} = \lambda_{2i+2}$, then there are two linearly independent eigenfunctions $\psi_{2i+1}$, $\psi_{2i+2}$ at $\lambda = \lambda_{2i+1} = \lambda_{2i+2}$, and consequently, all solutions of* (1.1) *are $T$-periodic. Similar results hold for the cases $\widetilde{\lambda}_{2i+1} < \widetilde{\lambda}_{2i+2}$ and $\widetilde{\lambda}_{2i+1} = \widetilde{\lambda}_{2i+2}$; let $\widetilde{\psi}_{2i+1}$, $\widetilde{\psi}_{2i+2}$ denote the corresponding eigenfunctions. Furthermore, $\psi_0$ has no zero on $[0, T]$; $\psi_{2i+1}$, $\psi_{2i+2}$ ($i \ge 0$) each have exactly $2i + 2$ zeros in $[0, T)$, and $\widetilde{\psi}_{2i+1}$, $\widetilde{\psi}_{2i+2}$ each have exactly $2i + 1$ zeros in $[0, T)$.*

The importance of the theorem can be illuminated by the fact that the special case of

$$P(t) \equiv 1 \quad (0 \le t < 2\pi), \qquad P \text{ is } 2\pi\text{-periodic},$$

yields the eigenvalues

$$\lambda_0 = 0,\ \lambda_{2i+1} = \lambda_{2i+2} = i + 1 \quad (i = 0, 1, \dots)$$

of the first kind and the well-known trigonometric system

$$\psi_0(t) = 1,\ \psi_{2i+1}(t) = \sin((i + 1)t),\ \psi_{2i+2}(t) = \cos((i + 1)t), \quad (i = 0, 1, \dots),$$

which is the main tool of the Fourier analysis and the functional analysis.

E. Meissner [15] studied the case of (1.1) when the coefficient $P$ is a piece-wise constant function assuming two different values in the interval $[0, T]$. This case is of special interest in technical applications and control due to, among others, the bang-bang principle [2]. As H. Hochstadt proved in [12], this case is also important because then the eigenvalues can be expressed by elementary functions.

The common starting points in all of the above mentioned studies were the Floquet Theory for the linear equations with periodic coefficients. For equation (1.1), after the introduction of the state variable vector $(x, x')$ and rewriting (1.1) into a two-dimensional system of linear differential equations, it says that the fundamental matrix $F\colon [0, \infty) \to \mathbb{C}^{2\times 2}$ of (1.1) with $F(0) = E$ ($E$ denotes the unit matrix in $\mathbb{C}^{2\times 2}$) has the representation

$$F(t) = S(t)e^{Rt} \quad (t \geq 0),$$

where $S$ is a non-singular $T$-periodic matrix function and $R \in \mathbb{C}^{2\times 2}$ is a constant matrix [3]. This makes it possible to reduce the problem of the existence of $T$-periodic and $2T$-periodic solutions (and the stability problems of the zero solution) to the study of the eigenvalues of the so-called monodromy matrix $C = e^{RT}$. Namely, (1.1) has a nontrivial $T$-periodic (respectively, $2T$-periodic) solution if and only if 1 (respectively, $-1$) is an eigenvalue of $C$.

Hochstadt [12] considered the Meissner equation

$$x'' + \lambda^2 Q(t)x = 0, \tag{1.3}$$

where $\lambda > 0$ is a real parameter;

$$Q(t) = \begin{cases} 1, & \text{if } 0 \leq t < 2, \\ a^2, & \text{if } 2 \leq t < 2L, \end{cases}$$

and $Q$ is to be continued as a $2L$-periodic function; $a$, $L$ ($0 < a \neq 1$, $L > 1$) are given constants. He computed the monodromy matrix $C$ and gave conditions for the existence of $2L$- and $4L$-periodic solutions for (1.3) in the form of equations for $\lambda$ with parameters $a$ and $L$, containing only elementary (trigonometric) functions.

In this paper we suggest a new method to study Meissner equations and apply the method to proving the *complete* Theorem A for equation (1.3). For the eigenvalues we get the same equations deduced by Hochstadt by Floquet Theory. What is more, to every eigenvalue we also *construct the corresponding eigenfunction(s).* The construction makes it possible to determine the number of zeros of these functions in the interval $[0, 2L)$, that cannot be done by Hochstadt's approach. So we can prove not only the distribution (1.2) of the eigenvalues but the whole oscillation theorem including the statement about the number of the linearly independent periodic solutions and about the number of the zeros of these solutions. We emphasize that our approach is graphic, constructive, and purely elementary not using either Floquet Theory. It is based only upon some properties of some simple geometric transformations on the plane $\mathbb{R}^2$.

It is interesting that the Meissner equation (1.3) is a very special case of the Hill's equation (1.1), nevertheless, the system of the eigenvalues and the eigenfunctions shows every feature of that of the most general case. Giving an entirely elementary proof for the *complete* Oscillation Theorem in this special case, we try to visualize this central theorem of differential equations also for students familiar only with elementary calculus.

In Section 2 we establish our method to study linear differential equations with step function coefficients. In Section 3 we apply the method to the deduction of Hochstadt's equation for the eigenvalues of (1.3) without Floquet Theory and without solving trigonometric equations. In Section 4 we complete the proof of Theorem A for the equation (1.3).

## 2. The method

We treat general (not only periodic) second order linear differential equations with step function coefficients. Given sequences $\{a_k\}_{k=1}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$ ($a_k > 0$; $t_0 = 0$, $t_k \nearrow \infty$ as $k \to \infty$), consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \text{ if } t_{k-1} \le t < t_k \qquad (k \in \mathbb{N}). \tag{2.1}$$

By a *solution* of this equation we mean a function $x : [0, \infty) \to \mathbb{R}$ that is continuously differentiable in $[0, \infty)$, twice differentiable and solves the equation for all $t \neq t_k$ ($k \in \mathbb{N}$). Studying (2.1) one usually introduces the variables $x, y := x'$, solves the corresponding system on the intervals $[t_{k-1}, t_k)$, then "glues the pieces together" so that $x$ and $y$ be continuous on the whole line $[0, \infty)$ [1], [5], [6], [12],

[16]. We suggest another method, which transforms (2.1) into an equivalent *discrete* dynamical system on the plane (see also [7], [8], [4]).

Let us introduce the state variables $x$, $y := x'/a_k$ provided that $t_{k-1} \leq t < t_k$. With these new variables, the second order differential equation (2.1) has the form of the 2-dimensional system of first order differential equations

$$x' = a_k y, \; y' = -a_k x \qquad \text{if } t_{k-1} \leq t < t_k \quad (k \in \mathbb{N}). \tag{2.2}$$

However, the collection of these systems ($k \in \mathbb{N}$) will be equivalent to equation (2.1) only if we guarantee the continuity of the functions $t \mapsto x(t)$, $t \mapsto x'(t)$ for *all* $t \geq 0$. To this end it is enough to have $x(t_k) = x(t_k - 0)$, $x'(t_k) = x'(t_k - 0)$ ($k \in \mathbb{N}$), where $f(t-0)$ denotes the left-hand side limit of function $f$ at $t$. Let the first equality be required as an initial condition on the interval $[t_k, t_{k+1})$. Expressing the second one by variable $y$ we get $a_{k+1} y(t_k) = a_k y(t_k - 0)$ for every $k \in \mathbb{N}$, which yields the other initial condition on $[t_k, t_{k+1})$. This means that (2.2) is equivalent to the system with impulses

$$\begin{cases} x' = a_k y, \quad y' = -a_k x, & \text{if } t_{k-1} \leq t < t_k, \\ x(t_k) = x(t_k - 0), \quad y(t_k) = \frac{a_k}{a_{k+1}} y(t_k - 0) & (k \in \mathbb{N}). \end{cases} \tag{2.3}$$

Given a pair $x_0$, $y_0$, we construct the solution of (2.3) on $[0, \infty)$ satisfying the initial condition $x(t_0) = x_0$, $y(t_0) = y_0$ in the following way. We solve equation (2.2) with these initial conditions in $[t_0, t_1)$ and get the solution $x_1, \; y_1 : [t_0, t_1) \to \mathbb{R}$. Then we define $x_2(t_1) := x_1(t_1 - 0)$, $y_2(t_1) := (a_1/a_2) y_1(t_1 - 0)$, and solve equation (2.2) with these initial conditions in $[t_1, t_2)$, and so on. The function

$$x(t) := x_k(t) \quad \text{if } t_{k-1} \leq t < t_k \qquad (k \in \mathbb{N})$$

will be the solution of (2.1) on $[0, \infty)$ satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = y_0$.

System of equations (2.3) is essentially equivalent to a very simple discrete dynamical system on the plane. In fact, if $x_k$, $y_k : [t_{k-1}, t_k) \to \mathbb{R}$ is a solution of (2.2), then

$$(x_k^2(t) + y_k^2(t))' = 2a_k(x_k(t) y_k(t) - y_k(t) x_k(t)) \equiv 0 \qquad (t_{k-1} \leq t < t_k),$$

which means that the trajectory of this solution is located on a circle around the origin. Introduce the polar coordinates $r, \varphi$ on the plane $x, y$ by

$$x = r \cos \varphi, \quad y = r \sin \varphi \qquad (r > 0, -\infty < \varphi < \infty). \tag{2.4}$$

We already know that $r'(t) = 0$ along any solution of (2.2) in every interval $[t_{k-1}, t_k)$. Since

$$x'(t) = -r(t)\varphi'(t)\sin\varphi(t) = a_k y(t) = a_k r(t)\sin\varphi(t) \qquad (t_{k-1} \le t < t_k),$$

we have

$$\varphi'(t) = -a_k \qquad (t_{k-1} \le t < t_k). \tag{2.5}$$

This means that the continuous components of the dynamics of (2.3) are uniform clockwise rotations around the origin with the angle velocity $a_k$. The impulsive steps of the dynamics, the "jumps", are either contractions or dilations in the direction of the $y$-axis. Introducing the notations $R_\theta$ and $C_\kappa$ for the matrices of the rotation and the contraction-dilation, respectively, i.e.,

$$\begin{aligned} R_\theta &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (-\infty < \theta < \infty), \\ C_\kappa &= \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} \qquad (0 < \kappa < \infty), \end{aligned} \tag{2.6}$$

we can rewrite system (2.3) into a discrete dynamical system on the plane $(\xi, \eta)^T$, whose trajectories are

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = \begin{pmatrix} x_{k+1}(t_k) \\ y_{k+1}(t_k) \end{pmatrix} \qquad (k = 0, 1, 2, \ldots).$$

In what follows we do not need functions $x_k, y_k$ any more, so, for the convenience, we can use the notation $(x_k, y_k)^T$ instead of $(\xi_k, \eta_k)^T$. Then the new form of (2.3) is

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = C_{a_{k+1}/a_{k+2}} R_{a_{k+1}(t_{k+1} - t_k)} \begin{pmatrix} x_k \\ y_k \end{pmatrix} \qquad (k = 0, 1, 2, \ldots). \tag{2.7}$$

Summing up, the steps of the dynamics of (2.3) can be described as follows. We start from a point $(x_0, y_0)$. We turn clockwise this point around the origin by $a_1(t_1 - t_0)$, then we apply a contraction or dilation of parameter $a_1/a_2$ parallel with the $y$-axis. This will be the map $(x_1, y_1)$ of the point $(x_0, y_0)$ after the first step. We apply the same two transformations on the new point $(x_1, y_1)$ with the new parameters $a_2(t_2 - t_1)$ and $a_2/a_3$. We repeat these steps ad infinitum (see Figure 1).

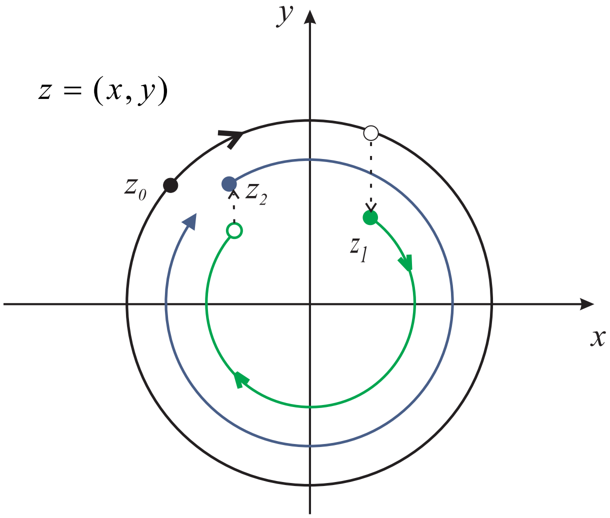


**Figure 1.** The dynamics of system (2.7)

Now let us switch over to polar coordinates in system (2.7). Denote by $(r_R, \varphi_R)$, and $(r_C, \varphi_C) = (\rho(r, \varphi; \kappa), \phi(\varphi; \kappa))$, the image of the point $(r, \varphi)$ in polar coordinates at the rotation, and the contraction-dilation (2.6), respectively. Then, obviously, $r_R(r, \varphi) = r$, $\varphi_R(r, \varphi) = \varphi - \theta$; furthermore,

$$
\begin{aligned}
\rho(r, \varphi; \kappa) &= \sqrt{x^2 + \kappa^2 y^2} = r\sqrt{1 + (\kappa^2 - 1)\sin^2\varphi} = f(\varphi; \kappa) r, \\
f(\varphi; \kappa) &:= \sqrt{1 + (\kappa^2 - 1)\sin^2\varphi} \qquad (\kappa > 0; -\infty < \varphi < \infty).
\end{aligned}
\tag{2.8}
$$

For $\phi(\varphi; \kappa)$, we know that $\tan\phi(\varphi; \kappa) = \kappa y/x = \kappa\tan\varphi$ for $x \neq 0$, consequently

$$
\phi(\varphi; \kappa) := \begin{cases} \arctan(\kappa\tan\varphi) + [\frac{\varphi + \pi/2}{\pi}]\pi, & \text{if } \varphi \neq (2k+1)\pi/2, \\ \varphi, & \text{if } \varphi = (2k+1)\pi/2, \end{cases} \qquad (k \in \mathbb{Z}),
\tag{2.9}
$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Now system (2.3) in polar coordinates has the form

$$
\begin{cases} r_{k+1} = f\Big(\varphi_k - a_{k+1}(t_{k+1} - t_k); \frac{a_{k+1}}{a_{k+2}}\Big) r_k, \\ \varphi_{k+1} = \phi\Big(\varphi_k - a_{k+1}(t_{k+1} - t_k); \frac{a_{k+1}}{a_{k+2}}\Big), \end{cases} \qquad (k = 0, 1, 2, \ldots).
\tag{2.10}
$$

The following lemma summarizes the basic properties of functions $f$ and $\phi$.

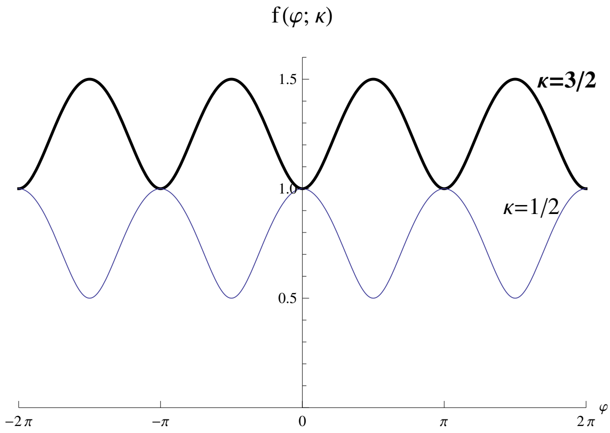


**Figure 2.** Graphs of functions $f(\cdot;\kappa)$ for $\kappa < 1$ and $\boldsymbol{\kappa} > \mathbf{1}$

**Lemma 2.1.** 1. *For every $\kappa > 0$ the function $f(\cdot;\kappa) : [0,\infty) \to (0,\infty)$ is even and $\pi$-periodic; furthermore,*

$$f\Big(\phi(\varphi;\kappa);\frac{1}{\kappa}\Big) = \frac{1}{f(\varphi;\kappa)} \quad (\varphi \in \mathbb{R}) \tag{2.11}$$

*(see Figure 2).*

2. *For every $\kappa > 0$ the functions $\phi(\cdot;\kappa)$ and $\phi(\cdot + \pi/2;\kappa) - \pi/2$ are odd, $\phi(\cdot + k\pi;\kappa) = \phi(\cdot;\kappa) + k\pi$ $(k \in \mathbb{Z})$; furthermore,*

$$\phi\Big(\phi(\varphi;\kappa);\frac{1}{\kappa}\Big) = \varphi \quad (\varphi \in \mathbb{R}). \tag{2.12}$$

3. *If $0 < \kappa < 1$, then for all $k \in \mathbb{Z}$ we have*

$$\begin{aligned} &\phi(\varphi;\kappa) < \varphi, \quad \text{if} \quad 2k\frac{\pi}{2} < \varphi < (2k+1)\frac{\pi}{2}, \\ &\phi(\varphi;\kappa) > \varphi, \quad \text{if} \quad (2k+1)\frac{\pi}{2} < \varphi < 2(k+1)\frac{\pi}{2}. \end{aligned} \tag{2.13}$$

*If $\kappa > 1$, then the inequalities between $\phi(\varphi;\kappa)$ and $\varphi$ are of the opposite directions (see Figure 3).*

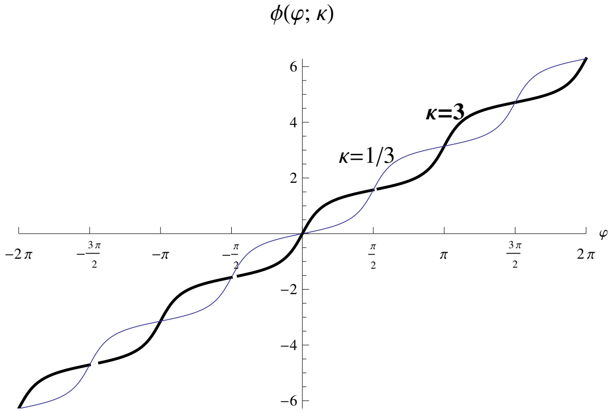


**Figure 3.** Graphs of functions $\phi(\cdot;\kappa)$ for $\kappa<1$ and $\boldsymbol{\kappa>1}$

**Proof.** Formulae (2.11), (2.12) are consequences of the identity $C_\kappa^{-1}=C_{1/\kappa}$; the other properties are obvious. ∎

## 3. The eigenvalues for equation (1.3)

It is clear that $\lambda_0=0$ is an eigenvalue with the eigenfunction $\psi_0(t)\equiv 1$.

**Theorem 3.1.** *For $\lambda>0$ equation (1.3) has a $2L$-periodic solution if and only if $\lambda$ is a solution either of equation*

$$-\phi\Big(\lambda;\frac{1}{a}\Big)+(m+1)\pi=a(L-1)\lambda, \tag{3.1}$$

*or of equation*

$$\Big(-\phi\Big(\lambda+\frac{\pi}{2};\frac{1}{a}\Big)+\frac{\pi}{2}\Big)+(m+1)\pi=a(L-1)\lambda \tag{3.2}$$

*for some integer $m\geq 0$.*

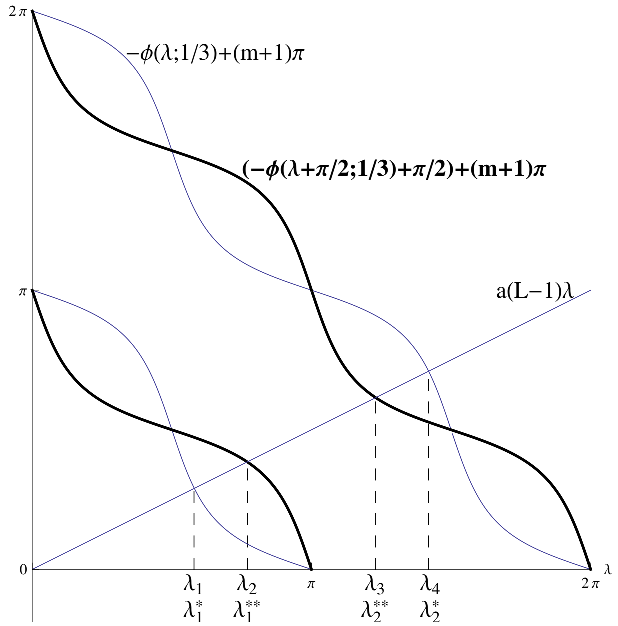


**Figure 4.** The eigenvalues of the first kind (see Theorem 3.1)

**Theorem 3.2.** *For $\lambda > 0$ equation (1.3) has a $4L$-periodic solution if and only if $\lambda$ is a solution either of equation*

$$-\phi\Big(\lambda; \frac{1}{a}\Big) + (2m+1)\frac{\pi}{2} = a(L-1)\lambda, \tag{3.3}$$

*or of equation*

$$\Big(-\phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) + \frac{\pi}{2}\Big) + (2m+1)\frac{\pi}{2} = a(L-1)\lambda \tag{3.4}$$

*for some integer $m \geq 0$.*

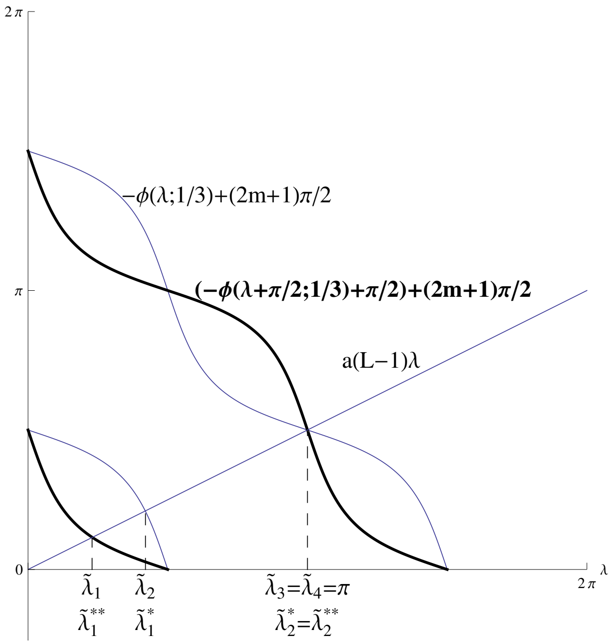


**Figure 5.** The eigenvalues of the second kind (see Theorem 3.2)

**Proof of Theorem 3.1.** For equation (1.3) the equivalent difference equation (2.7) has the form

$$\text{(3.5)} \qquad \begin{cases} \begin{pmatrix} x_{2\ell+1} \\ y_{2\ell+1} \end{pmatrix} = C_{1/a} R_{2\lambda} \begin{pmatrix} x_{2\ell} \\ y_{2\ell} \end{pmatrix}, \\ \begin{pmatrix} x_{2\ell+2} \\ y_{2\ell+2} \end{pmatrix} = C_a R_{2\lambda a(L-1)} \begin{pmatrix} x_{2\ell+1} \\ y_{2\ell+1} \end{pmatrix} \end{cases} \qquad (l = 0, 1, 2, \dots).$$

We introduce the notations

$$\text{(3.6)} \qquad \begin{array}{lll} r_0 := r(0), & \varphi_0 :\equiv \varphi(0) \pmod{2\pi}, & -\pi \le \varphi_0 < \pi; \\ r_1 := r(2-0)(= r_0), & \varphi_1 := \varphi(2-0) & \\ r_2 := r(2) = f(\varphi_1; 1/a) r_1, & \varphi_2 := \varphi(2) = \phi(\varphi_1; 1/a); & \\ r_3 := r(2L-0)(= r_2), & \varphi_3 := \varphi(2L-0); & \\ r_4 := r(2L) = f(\varphi_3; a) r_3, & \varphi_4 := \varphi(2L) = \phi(\varphi_3; a), & \end{array}$$

and correspondingly,

$$x_i := r_i \cos \varphi_i, \quad y_i = r_i \sin \varphi_i \quad (i = 0, 1, 2, 3, 4);$$

(the numberings in the indices of $r_i, \varphi_i$ and $x_i, y_i$ here differ from those of (2.10) and (3.5)) (see Figure 6).

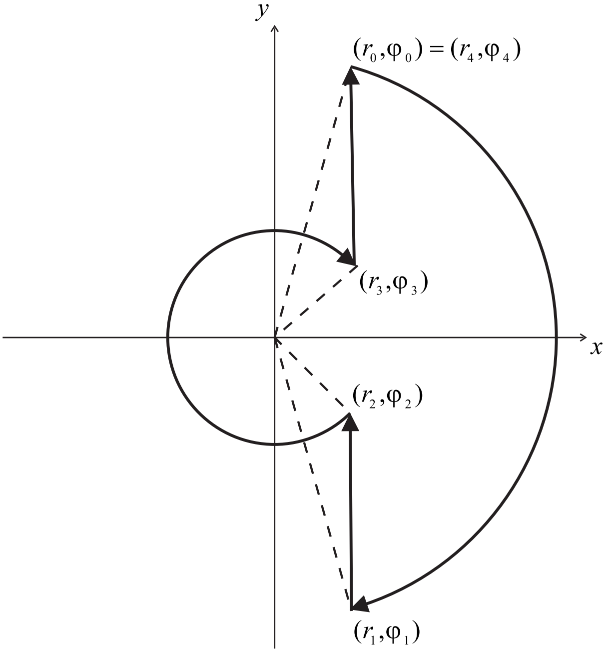


**Figure 6.** The 4 steps of the dynamics during one period of a $2L$-periodic solution

*Necessity.* Suppose that $t \mapsto (r(t), \varphi(t))$ corresponds to a $2L$-periodic solution of (1.3). Then $y_4 = ay_3$, i.e., $y_3 = y_4/a$, and $x_3 = x_4$, so we have

$$r_3 = f\Big(\varphi_4; \frac{1}{a}\Big) r_4, \quad \varphi_3 = \phi\Big(\varphi_4; \frac{1}{a}\Big). \tag{3.7}$$

But $r_4 = r_0$, $\varphi_4 \equiv \varphi_0 \pmod{2\pi}$ because the solution is $2L$-periodic; therefore,

$$\begin{aligned} r_2 &= f\Big(\varphi_1; \frac{1}{a}\Big) r_1 = f\Big(\varphi_1; \frac{1}{a}\Big) r_0, \\ r_3 &= f\Big(\varphi_4; \frac{1}{a}\Big) r_4 = f\Big(\varphi_0; \frac{1}{a}\Big) r_0 \end{aligned} \tag{3.8}$$

since $f(\cdot; 1/a)$ is $\pi$-periodic. Moreover, $r_2 = r_3$, so we get

$$f\Big(\varphi_1; \frac{1}{a}\Big) = f\Big(\varphi_0; \frac{1}{a}\Big). \tag{3.9}$$

Function $f$ is even and $\pi$-periodic, thus this equality implies that either

(a) $\varphi_1 \equiv -\varphi_0 \pmod{\pi}$,

or

(b) $\varphi_1 \equiv \varphi_0 \pmod{\pi}$.

In the case (a) there are two possibilities:

(a.1) $\varphi_1 \equiv -\varphi_0 \pmod{2\pi}$,

(a.2) $\varphi_1 \equiv -(\varphi_0 + \pi) \pmod{2\pi}$.

In the case (a.1) there exists an integer $k > -\varphi_0/\pi$ such that $\varphi_1 = -\varphi_0 - 2k\pi$. By (2.5), $\dot{\varphi}(t) = -\lambda$ $(0 \le t < 2)$, so $\varphi_1 - \varphi_0 = -2\varphi_0 - 2k\pi = -2\lambda$, whence we obtain

$$\varphi_0 = \lambda - k\pi. \tag{3.10}$$

Let us express $\varphi_2$ with $\lambda$:

$$\begin{aligned}\varphi_2 &= \phi\Big(\varphi_1; \frac{1}{a}\Big) = \phi\Big(-\varphi_0 - 2k\pi; \frac{1}{a}\Big) = \phi\Big(-\lambda - k\pi;\ \frac{1}{a}\Big)\\ &= -\phi\Big(\lambda; \frac{1}{a}\Big) - k\pi.\end{aligned}$$

By (2.5) again, $\dot{\varphi}(t) = -a\lambda$ $(2 \le t < 2L)$; therefore, using also (3.7)–(3.10), we have

$$\begin{aligned}\varphi_3 &= \phi\Big(\varphi_0; \frac{1}{a}\Big) - 2(m+1)\pi = \phi\Big(\lambda; \frac{1}{a}\Big) - (2(m+1)+k)\pi,\\ \varphi_3 - \varphi_2 &= 2\phi\Big(\lambda; \frac{1}{a}\Big) - 2(m+1)\pi = -2a(L-1)\lambda\end{aligned} \tag{3.11}$$

for some integer $m \ge 0$, i.e., $\lambda$ satisfies (3.1).

Similarly, in the case (a.2) there exists an integer $k > -\varphi_0/\pi - 1/2$ such that $\varphi_1 = -\varphi_0 - (2k+1)\pi$, and $\varphi_1 - \varphi_0 = -2\varphi_0 - (2k+1)\pi = -2\lambda$, consequently,

$$\varphi_0 = \lambda - \frac{2k+1}{2}\pi. \tag{3.12}$$

Then

$$\begin{aligned}\varphi_2 =& \phi\Big(\varphi_1; \frac{1}{a}\Big) = \phi\Big(-\varphi_0 - (2k+1)\pi; \frac{1}{a}\Big) = \phi\Big(-\lambda - \frac{2k+1}{2}\pi; \frac{1}{a}\Big)\\ =& -\phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - k\pi,\\ \varphi_3 =& \phi\Big(\varphi_0; \frac{1}{a}\Big) - 2(m+1)\pi = \phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - (2(m+1)+k+1)\pi,\\ \varphi_3 - \varphi_2 =& 2\phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - \pi - 2(m+1)\pi = -2a(L-1)\lambda \qquad (3.13)\end{aligned}$$

for some integer $m \ge 0$, i.e., $\lambda$ satisfies (3.2).

If (b) is satisfied, then there is an integer $k \ge 0$ such that $\varphi_1 - \varphi_0 = -(k+1)\pi = -2\lambda$, i.e., $\lambda = (k+1)\pi/2$, and

$$\begin{aligned}\varphi_3 - \varphi_2 &= \Big(\phi\Big(\varphi_0; \frac{1}{a}\Big) - 2(m+1)\pi\Big) - \phi\Big(\varphi_0 - (k+1)\pi; \frac{1}{a}\Big) \\ &= -(2(m+1) - (k+1))\pi = -2a(L-1)\lambda\end{aligned} \tag{3.14}$$

with some integer $m \ge 0$. This means that in the case (b) there are integers $k, m \ge 0$ such that

$$\lambda = (k+1)\frac{\pi}{2}, \quad \Big(\frac{2(m+1)}{2} - \frac{k+1}{2}\Big)\pi = a(L-1)\lambda. \tag{3.15}$$

It is easy to see that (3.15) implies both (3.1) and (3.2).

*Sufficiency.* Suppose that $\lambda$ satisfies (3.1) for some integer $m \ge 0$. This condition was deduced from the case (a.1), so it is reasonable to define

$$\varphi_0 := \Big\{\frac{\lambda}{\pi}\Big\}\pi \tag{3.16}$$

(see (3.10)), where $\{x\}$ denotes the fractional part of $x$. We show that the solution $t \mapsto (r(t), \varphi(t))$ satisfying the initial conditions $r(0) = 1$, $\varphi(0) = \varphi_0$ is $2L$-periodic. Using notations (3.6), the expression

$$\lambda = \Big\{\frac{\lambda}{\pi}\Big\}\pi + \Big[\frac{\lambda}{\pi}\Big]\pi = \varphi_0 + \Big[\frac{\lambda}{\pi}\Big]\pi, \tag{3.17}$$

and the properties of $f$, $\phi$ summarized in Lemma 2.1, we get

$$\begin{aligned}\varphi_2 &= \phi\Big(\varphi_1; \frac{1}{a}\Big) = \phi\Big(\varphi_0 - 2\lambda; \frac{1}{a}\Big) = -\phi\Big(\varphi_0; \frac{1}{a}\Big) - 2\Big[\frac{\lambda}{\pi}\Big]\pi, \\ \varphi_3 &= \varphi_2 - 2a(L-1)\lambda = \varphi_2 + 2\phi\Big(\varphi_0; \frac{1}{a}\Big) + 2\Big[\frac{\lambda}{\pi}\Big]\pi - 2(m+1)\pi \\ &= \phi\Big(\varphi_0; \frac{1}{a}\Big) - 2(m+1)\pi,\end{aligned} \tag{3.18}$$

$$\varphi_4 = \phi(\varphi_3; a) = \phi\Big(\phi\Big(\varphi_0; \frac{1}{a}\Big); a\Big) - 2(m+1)\pi = \varphi_0 - 2(m+1)\pi. \tag{3.19}$$

On the other hand

$$\begin{aligned}r_3 = r_2 &= f\Big(\varphi_1; \frac{1}{a}\Big)r_1 = f\Big(\varphi_0 - 2\lambda; \frac{1}{a}\Big)r_0 \\ &= f\Big(\varphi_0 - 2\Big[\frac{\lambda}{\pi}\Big]\pi - 2\varphi_0; \frac{1}{a}\Big)r_0 = f\Big(\varphi_0; \frac{1}{a}\Big)r_0, \\ r_4 &= f(\varphi_3; a)r_3 = f\Big(\phi\Big(\varphi_0; \frac{1}{a}\Big) - 2(m+1)\pi; a\Big)r_3 \\ &= f\Big(\phi\Big(\varphi_0; \frac{1}{a}\Big); a\Big) f\Big(\varphi_0; \frac{1}{a}\Big)r_0 = r_0,\end{aligned} \tag{3.20}$$

which, together with $\varphi_4 \equiv \varphi_0 \pmod{2\pi}$, means that the solution is $2L$-periodic.

Now suppose that $\lambda$ satisfies (3.2). This condition comes from the case (a.2), so (3.12), i.e., $\varphi_0 + \pi/2 = \lambda - k\pi$ suggests choosing

$$\varphi_0 := \Big\{\frac{\lambda}{\pi}\Big\}\pi - \pi/2. \tag{3.21}$$

Then

$$\lambda = \Big\{\frac{\lambda}{\pi}\Big\}\pi + \Big[\frac{\lambda}{\pi}\Big]\pi = \varphi_0 + \frac{\pi}{2} + \Big[\frac{\lambda}{\pi}\Big]\pi. \tag{3.22}$$

A simple repetition of the computations above with (3.2), (3.22) instead of (3.1), (3.17) yields the relations

$$\varphi_4 = \varphi_0 - 2(m+1)\pi, \quad r_4 = r_0, \tag{3.23}$$

guaranteeing that the solution is $2L$-periodic.

■

**Proof of Theorem 3.2.** *Necessity.* Suppose that $t \mapsto (r(t), \varphi(t))$ is a $4L$-periodic solution of (3.5). In the language of the notation system (3.6) introduced at the beginning of the proof of the previous theorem we can say that the solution is $4L$-periodic if and only if $r_4 = r_0$, $\varphi_4 \equiv \varphi_0 - \pi \pmod{2\pi}$. Since $f$ is $\pi$-periodic, (3.8) and (3.9) remain valid without any modification, so we have to distinguish the same cases (a.1), (a.2), (b) even now. On the other hand, in virtue of $\varphi_4 \equiv \varphi_0 - \pi$ (mod $2\pi$), there exists an integer $m \geq 0$ such that $\varphi_4 = \varphi_0 - (2m+1)\pi$, and

$$\varphi_3 = \phi\Big(\varphi_4; \frac{1}{a}\Big) = \phi\Big(\varphi_0; \frac{1}{a}\Big) - (2m+1)\pi.$$

So, if we repeat the computations made in the "Necessity" part of the previous proof, we get formulae (3.11), (3.13)–(3.15) with the only modifications that $2(m+1)$ *everywhere should be changed to* $(2m+1)$. This means that in the different cases we get the following equations for $\lambda$:

(a.1): $\varphi_3 - \varphi_2 = 2\phi(\lambda; 1/a) - (2m+1)\pi = -2a(L-1)\lambda$,
(a.2): $\varphi_3 - \varphi_2 = 2\phi(\lambda + \frac{\pi}{2}; 1/a) - \pi - (2m+1)\pi = -2a(L-1)\lambda$,
(b): $\lambda = (k+1)\frac{\pi}{2}$, $(\frac{2m+1}{2} - \frac{k+1}{2})\pi = a(L-1)\lambda$.

Consequently, either (3.3) or (3.4) is satisfied. In case (b) both (3.3) and (3.4) hold.

*Sufficiency.* We choose the same initial conditions as in the proof of the previous theorem. For $\varphi_3$ and $\varphi_4$ we get the same formulae (3.18), (3.19) with the only difference that $(2m+1)$ stands instead of $2(m+1)$, so we come to $\varphi_4 \equiv \varphi_0 - \pi$ (mod $2\pi$) and $r_4 = r_0$, which means that the solution is $4L$-periodic.

■

## 4. The numbers of the linearly independent periodic solutions and their zeros

First of all, it is worth clearing up: when does $\lambda$ satisfy both (3.1) and (3.2)?

**Lemma 4.1.** *$\lambda > 0$ satisfies both (3.1) and (3.2) with the same integer $m \geq 0$ if and only if $\lambda = (k+1)\pi/2$ with some integer $k \geq 0$. The same is true for the equalities (3.3) and (3.4)*

**Proof.** It is enough to prove the first assertion. By (2.9), $\phi(l\pi/2; 1/a) = l\pi/2$ ($l \in \mathbb{Z}$), so the "if" part is obvious. To prove the "only if" part, suppose that both (3.1) and (3.2) are satisfied with the same $m \geq \mathbb{Z}$. Then

$$\phi\Big(\lambda; \frac{1}{a}\Big) = \phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - \frac{\pi}{2}. \tag{4.1}$$

Let, e.g., $a > 1$. By (2.13), for all $\ell \in \mathbb{Z}$ we have the inequalities

$$\phi\Big(\lambda; \frac{1}{a}\Big) < \lambda, \quad \phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - \frac{\pi}{2} > \lambda \qquad \Big(2\ell\frac{\pi}{2} < \lambda < (2\ell+1)\frac{\pi}{2}\Big),$$
$$\phi\Big(\lambda; \frac{1}{a}\Big) > \lambda, \quad \phi\Big(\lambda + \frac{\pi}{2}; \frac{1}{a}\Big) - \frac{\pi}{2} < \lambda \qquad \Big((2\ell+1)\frac{\pi}{2} < \lambda < 2(\ell+1)\frac{\pi}{2}\Big);$$

therefore, (4.1) implies $\lambda = (k+1)\pi/2$ for some integer $k \geq 0$. Similarly, this is also true if $a > 1$, i.e., (4.1) implies $\lambda = (k+1)\pi/2$.

■

Now we complete the proof of Theorem A for equation (1.3) counting the linearly independent eigenfunctions and their zeros.

To the eigenvalue $\lambda_0 = 0$ there belongs the unique eigenfunction $\psi_0(t) \equiv 1$. (As it was mentioned earlier, two eigenfunctions are distinguished only if they are linearly independent.)

Functions on the left-hand sides of (3.1), (3.2) are strictly decreasing, continuous, and their graphs cross the positive half-axes on the plane $\lambda, \phi$, so for every $m \geq 0$ there exist exactly one $\lambda_m^* > 0$ and exactly one $\lambda_m^{**} > 0$ such that $\lambda_m^*$ satisfies (3.1), and $\lambda_m^{**}$ satisfies (3.2) (see Figure 4). By Theorem 3.1, $\lambda_m^*$, $\lambda_m^{**}$ are eigenvalues of the first kind of equation (1.3). Let $x = \psi_m^*$ and $x = \psi_m^{**}$ denote the corresponding eigenfunctions satisfying the following initial conditions in polar coordinates $r, \varphi$:

$$\psi_m^* : \; r_m^*(0) = 1, \quad \varphi_m^*(0) = \Big\{\frac{\lambda_m^*}{\pi}\Big\}\pi,$$
$$\psi_m^{**} : \; r_m^{**}(0) = 1, \quad \varphi_m^{**}(0) = \Big\{\frac{\lambda_m^{**}}{\pi}\Big\}\pi - \frac{\pi}{2}$$

(see (3.16), (3.21)). If $\lambda_m^*$ and $\lambda_m^{**}$ are different, then by the "Necessity" part of the proof of Theorem 3.1 and formulae (3.10), (3.12), $\psi_m^*$ (respectively, $\psi_m^{**}$) is the unique eigenfunction to $\lambda_m^*$ (respectively, $\lambda_m^{**}$). If $\lambda_m^* = \lambda_m^{**} = (k+1)\pi/2$ with some integer $k \geq 0$, then $\psi_m^*$ and $\psi_m^{**}$ are two linearly independent $2L$-periodic solutions of the same equation, so $\lambda_m^* = \lambda_m^{**} = (k+1)\pi/2$ is an eigenvalue of the first kind of (1.3) with two linearly independent eigenfunctions. (In this case two periodic solutions are said to coexist [12].)

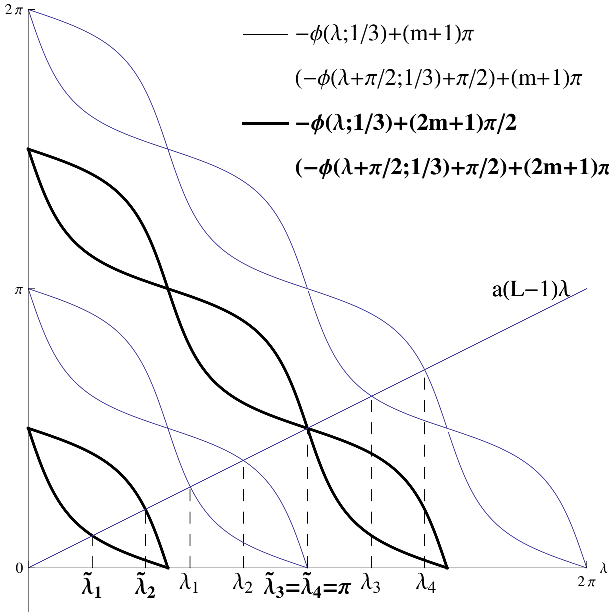


**Figure 7.** All the eigenvalues of equation (1.3) (see Theorem A)

Now we count the zeros of $\psi_m^*$ on $[0, 2L)$. As (2.4) shows, $\bar{t}$ is a zero of $\psi_m^*$ if and only if $\cos\varphi_m^*(\bar{t}) = 0$. By (3.19), the phase point $(\psi_m^*(t), y_m^*(t))$ turns around the origin on the $(x, y)$ plane $(m+1)$ times. Since $\varphi_m^*$ is strictly decreasing, this means that $\psi_m^*$ has exactly $2(m+1)$ zeros on $[0, 2L)$. In virtue of (3.23), the same is true for $\psi_m^{**}$.

Let us turn to the eigenvalues of the second kind. Using equations (3.3)–(3.4) instead of (3.1)–(3.2), we analogously introduce the eigenvalues $\widetilde{\lambda}_m^*$, $\widetilde{\lambda}_m^{**}$ (see Figure 5) and the corresponding eigenfunctions $x = \widetilde{\psi}_m^*$, $x = \widetilde{\psi}_m^{**}$ satisfying the initial conditions

$$\widetilde{\psi}_m^* : \widetilde{r}_m^*(0) = 1, \quad \widetilde{\varphi}_m^*(0) = \Big\{\frac{\widetilde{\lambda}_m^*}{\pi}\Big\}\pi,$$
$$\widetilde{\psi}_m^{**} : \widetilde{r}_m^{**}(0) = 1, \quad \widetilde{\varphi}_m^{**}(0) = \Big\{\frac{\widetilde{\lambda}_m^{**}}{\pi}\Big\}\pi - \frac{\pi}{2}.$$

If $\widetilde{\lambda}_m^*$ and $\widetilde{\lambda}_m^{**}$ are different, then $\widetilde{\psi}_m^*$ (respectively, $\widetilde{\psi}_m^{**}$) is the unique eigenfunction to $\widetilde{\lambda}_m^*$ (respectively, $\widetilde{\lambda}_m^{**}$). If $\widetilde{\lambda}_m^* = \widetilde{\lambda}_m^{**}$, then $\widetilde{\psi}_m^*$ and $\widetilde{\psi}_m^{**}$ are two linearly independent eigenfunctions to this eigenvalue of the second type.

By the teaching of the proof of Theorem 3.2, formulae (3.19) and (3.23) remain true for $\widetilde{\psi}_m^*, \widetilde{\psi}_m^{**}$ with $(2m+1)$ instead of $2(m+1)$. Consequently, the phase points $(\widetilde{\psi}_m^*(t), \widetilde{y}_m^*(t))$ and $(\widetilde{\psi}_m^{**}(t), \widetilde{y}_m^{**}(t))$ turn around the origin on the $(x, y)$ plane $m$ and a half times, which means that $\widetilde{\psi}_m^*$ and $\widetilde{\psi}_m^{**}$ each have exactly $(2m+1)$ zeros on $0, 2L)$.

Finally, let us rename $\lambda_m^*$, $\lambda_m^{**}$ (respectively, $\widetilde{\lambda}_m^*$, $\widetilde{\lambda}_m^{**}$) by $\lambda_{2m+1}$, $\lambda_{2m+2}$ (respectively, $\widetilde{\lambda}_{2m+1}, \widetilde{\lambda}_{2m+2}$) so that

$$\widetilde{\lambda}_{2m+1} \le \widetilde{\lambda}_{2m+2} < \lambda_{2m+1} \le \lambda_{2m+2}$$

be satisfied (see Figure 7). Then (1.2) is satisfied, and all assertions of Theorem A have been proved for equation (1.3).

## References


[1] V. I. Arnold, *Ordinary differential equations*, Universitext, Springer-Verlag, Berlin, 2006.

[2] M. Athans and P. L. Falb, *Optimal control. An introduction to the theory and its applications*, McGraw-Hill Book Co., New York – Toronto – London, 1966.

[3] E. A. Coddington and M. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York – Toronto – London, 1955.

[4] S. Csörgő and L. Hatvani, Stability properties of solutions of linear second order differential equations with random coefficients, *J. Differential Equations*, **248** (2010), 21–49.

[5] Á. Elbert, Stability of some difference equations, *Advances in difference equations* (Veszprém, 1995), Gordon and Breach, Amsterdam, 1997, 165–187.

[6] Á. Elbert, On damping of linear oscillators, *Studia Sci. Math. Hungar.*, **38** (2001), 191–208.

[7] L. Hatvani, On the existence of a small solution to linear second order differential equations with step function coefficients, *Dynam. Contin. Discrete Impuls. Systems*, **4** (1998), 321–330.

[8] L. Hatvani, On the critical values of parametric resonance in Meissner's equation by the method of difference equations, *Electron. J. Qual. Theory Differ. Equ. (Special Edition I)* (2009), No. 13, 10 pp.

[9] O. Haupt, Über eine Methode zum Bewise von Oszillationstheoremen, *Math. Ann.*, **76** (1914), 67–104.

[10] O. Haupt, Über lineare homogene Differentialgleichungen 2. Ordnung mit periodischen Koeffizienten, *Math. Ann.*, **79** (1917), 278–285.

[11] G. W. Hill, On the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon, *Acta Mathematica*, **8** (1886), 1–36.

[12] H. Hochstadt, A special Hill's equation with discontinuous coefficients, *Amer. Math. Monthly*, **70** (1963), 18–26.

[13] A. M. Lyapunov, The general problem of the stability of motion, *Internat. J. Control* (Lyapunov centenary issue), **55** (1992), 521–790.

[14] W. Magnus and S. Winkler, *Hill's equation*, Dover Publications, Inc., New York, 1979.

[15] E. Meissner, Über Schüttelschwingungen in Systemen mit periodisch veränderlicher Elastizität, *Schweizer Bauzeitung*, **72** (1918), 95–98.

[16] P. Pucci and J. Serrin, Asymptotic stability for intermittently controlled nonlinear oscillators, *SIAM J. Math. Anal.*, **25** (1994), 815–835.

L. Hatvani, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary; *e-mail*: hatvani@math.u-szeged.hu