doi: 10.14232/actasm-012-299-2

Acta Sci. Math. (Szeged)
80 (2014), 27–29

A short remark to an important inequality of Leindler

X. X. Xia and S. P. Zhou

Communicated by L. Kérchy

Abstract. This short note gives a mending to a little but sensitive flaw in the original proof of an important and useful inequality established by Leinder.

In 1970, Leindler [3] generalized some important inequalities of Hardy and Littlewood, among those the following two are the most useful (for convenience we only mention the standard version here):

Theorem. ([3]) *Let $p \geqslant 1$, $\alpha_n \geqslant 0$, then for all $\lambda_n > 0$ it holds that*

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=1}^{n} \alpha_k \Big)^p &\leqslant p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=n}^{\infty} \lambda_k \Big)^p \alpha_n^p, \\ \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=n}^{\infty} \alpha_k \Big)^p &\leqslant p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=1}^{n} \lambda_k \Big)^p \alpha_n^p. \end{aligned} \tag{1}$$

These two inequalities indeed have many important and useful applications, especially in L^p integrability of trigonometric series (cf. references [2, 4, 5], for example). Generalizations of these inequalities could be found in the work [6].

Let

$$A_n^* = \sum_{k=n}^{\infty} \alpha_k, \tag{2}$$

and

$$\Lambda_n = \sum_{k=1}^{n} \lambda_k, \quad n = 1, 2, \ldots. \tag{3}$$

Received June 20, 2012, and in revised form August 1, 2012.
AMS Subject Classification: 26D15.
Key words and phrases: inequality, mending.

Recently, we found that, for $p > 1$, in the original proof of inequality (1) there is some little but sensitive flaw and it really needs to be mended. Actually, in the original proof, suppose first that the right side of the second inequality in (1) is finite, then in the following inequality ([3, page 282, line -4])

$$\sum_{n=1}^{k-1} \Lambda_n \alpha_n (A_n^*)^{p-1} \leqslant \Big(\sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_n^p \alpha_n^p \Big)^{1/p} \Big(\sum_{n=1}^{\infty} \lambda_n (A_n^*)^p \Big)^{(p-1)/p}$$

and in the following texts, the assumption that $\sum_{n=1}^{\infty} \lambda_n (A_n^*)^p < \infty$ is already automatically applied.

In this short note, we will prove a lemma to solve the confusion, and all the other procedures in the original proof can be kept unchanged.

Lemma. *Let $p > 1$, $\alpha_n \geqslant 0$, $\lambda_n > 0$, $n = 1, 2, \ldots$, and Λ_n, A_n^* be defined as in* (2) *and* (3). *If $\sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_n^p \alpha_n^p < \infty$, then $\sum_{n=1}^{\infty} \lambda_n (A_n^*)^p < \infty$.*

Proof. Suppose, to the contrary, that

$$\sum_{n=1}^{\infty} \lambda_n (A_n^*)^p = \infty. \tag{4}$$

Applying Hölder's inequality, one gets

$$(A_n^*)^p \leqslant \Big(\sum_{k=n}^{\infty} \lambda_k^{1-p} \Lambda_k^p \alpha_k^p \Big) \Big(\sum_{k=n}^{\infty} \lambda_k \Lambda_k^{p/(1-p)} \Big)^{p-1}. \tag{5}$$

We already know that, for any given $\sigma > 0$, the series $\sum_{n=1}^{\infty} \lambda_n / \Lambda_n^{1+\sigma}$ converges (this fact is already mentioned in Hardy–Littlewood–Pólya [1, Theorem 162]) and

$$\sum_{k=n+1}^{\infty} \frac{\lambda_k}{\Lambda_k^{1+\sigma}} = O(\Lambda_n^{-\sigma}). \tag{6}$$

Indeed, in view of

$$\frac{1}{\sigma} \Big(\frac{1}{\Lambda_{n-1}^{\sigma}} - \frac{1}{\Lambda_n^{\sigma}} \Big) = \frac{\lambda_n}{\bar{\Lambda}_n^{1+\sigma}}, \quad \text{where } \Lambda_{n-1} < \bar{\Lambda}_n < \Lambda_n,$$

we can immediately deduce (6).

Now we check by (5) and (6) that

$$\Lambda_n(A_n^*)^p \leqslant M(p)\Big(\sum_{k=n}^{\infty}\lambda_k^{1-p}\Lambda_k^p\alpha_k^p\Big)\Big(\lambda_n^{p-1}\Lambda_n^{-p}+\Big(\sum_{k=n+1}^{\infty}\frac{\lambda_k}{\Lambda_k^{1+1/(p-1)}}\Big)^{p-1}\Big)\Lambda_n$$
$$\leqslant M(p)\Big(\sum_{k=n}^{\infty}\lambda_k^{1-p}\Lambda_k^p\alpha_k^p\Big)(\lambda_n^{p-1}\Lambda_n^{-p}+\Lambda_n^{-1})\Lambda_n = o(1), \quad n\to\infty,$$

where $M(p)$ is a positive constant only depending upon p. From this point, applying Abel's transformation and Hölder's inequality, we have, under the assumption (4), that

$$\sum_{n=1}^{k}\lambda_n(A_n^*)^p \leqslant \sum_{n=1}^{k-1}\Lambda_n\alpha_n(A_n^*)^{p-1}+\Lambda_k(A_k^*)^p \leqslant M(p)\sum_{n=1}^{k-1}\Lambda_n\alpha_n(A_n^*)^{p-1}$$
$$\leqslant M(p)\Big(\sum_{n=1}^{\infty}\lambda_n^{1-p}\Lambda_n^p\alpha_n^p\Big)^{1/p}\Big(\sum_{n=1}^{k}\lambda_n(A_n^*)^p\Big)^{(p-1)/p},$$

or in other words, it follows that

$$\Big(\sum_{n=1}^{k}\lambda_n(A_n^*)^p\Big)^{1/p} \leqslant M(p)\Big(\sum_{n=1}^{\infty}\lambda_n^{1-p}\Lambda_n^p\alpha_n^p\Big)^{1/p}$$

holds for any sufficiently large k, thus obtain a contradiction to (4). ■

References

[1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge, 1952.

[2] R. J. Le and S. P. Zhou, A remark on "two-sided" monotonicity condition: an application to L^p convergence, *Acta Math. Hungar.*, **113** (2006), 159–169.

[3] L. Leindler, Generalization of inequalities of Hardy and Littlewood, *Acta Sci. Math. (Szeged)*, **31** (1970), 279–285.

[4] L. Leindler, Relations among Fourier coefficients and sum-functions, *Acta Math. Hungar.*, **104** (2004), 171–183.

[5] D. S. Yu, P. Zhou and S. P. Zhou, On L^p integrability and convergence of trigonometric series, *Studia Math.*, **182** (2007), 215–226.

[6] J. Németh, Generalizations of the Hardy–Littlewood inequality. II., *Acta Sci. Math. (Szeged)*, **35** (1973), 127–134.

X. X. Xia, Institute of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China

S. P. Zhou, Institute of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China; *e-mail*: songping.zhou@163.com