A polyhedron without diagonals.

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It is simple to prove that *the tetrahedron is the only polyhedron homeomorphic to the sphere and having the property that every two of its vertices are joined by an edge.* In fact, such a polyhedron must be triangle-faced; and if we denote by v the number of its vertices, then it has $\binom{v}{2}$ edges and $\frac{2}{3}\binom{v}{2}$ faces, so that the theorem of EULER gives

$$\frac{2}{3}\binom{v}{2}+v-\binom{v}{2}=2. \tag{1}$$

This equation furnishes $v=3$ or $v=4$; the first solution has no geometrical meaning and the second gives the tetrahedron.

The question arises, whether this proposition remains true or not by omitting the restriction concerning the topological type of the polyhedron. We shall give in this paper a negative answer to this question by showing the existence of a polyhedron homeomorphic to the torus with the property mentioned above.

For the case of the torus, we have to put 0 instead of 2 on the right side of (1) and we obtain from this equation $v=7$. We first shall draw the 7 vertices and the 21 edges of our polyhedron on the torus.

Let us represent the torus on a rectangle $ABCD$. The opposite points lying on the sides of this rectangle are the images of the same point of the torus. Let us take seven points $1, 2, 3, \dots 7$ in this order on the side AB; they appear naturally on the opposite side CD too. By drawing the straight segments joining the point 1 (on AB) to the points 3 and 4 (on CD), then those joining the point 2 (on AB) to the points 4 and 5 (on CD) and so on in the cyclic order of the vertices, these segments together with the segments of AB (and CD) joining two neighbouring vertices, form a system of lines containing 21 edges which joins every pair of the seven vertices by an edge and divides the torus represented by the rectangle $ABCD$ in 14 triangles. Table 1 enumerates the vertices of these 14 triangles.

Table 1.

126	235	356	346	467	237	267
156	245	124	134	137	457	157

We shall now construct a polyhedron which realizes this topological scheme.

Table 2 shows the coordinates of the seven vertices of our polyhedron in a rectangular system of coordinates. The values of a and b will be given later on. This system of vertices shows an axial symmetry with respect to the z-axis, 1 and 6, 2 and 5, 3 and 4 corresponding to each other. As table 1 shows, the faces of the polyhedron show the same symmetry. We have to choose the values a and b in the way that no pair of the faces intersect each other.

Table 2.

	x	y	z
1	-3	3	0
2	-3	-3	a
3	-1	-2	3
4	1	2	3
5	3	3	a
6	3	-3	0
7	0	0	b

In we put for a moment $a=0$ and $b=+\infty$, a short computation shows that the plane passing through the vertices 356 divides the space in two half-spaces in the way that the points 1 and 2 lie in the first half-space and the points 4 and 7 in the second. We can use for the abbreviation of this fact the symbol

$$12 \mid 356 \mid 47 \quad \ldots \text{A.}$$

We get similarly

$$\begin{array}{rcl} 125 \mid 346 \mid 7 & \ldots & \text{B} \\ 145 \mid 236 \mid 7 & \ldots & \text{C} \\ 145 \mid 237 \mid 6 & \ldots & \text{D} \\ 23 \mid 167 \mid 45 & \ldots & \text{E} \\ 36 \mid 257 \mid 14 & \ldots & \text{F.} \end{array}$$

These propositions remain valid even if we give to a a sufficiently small and to b a sufficiently large positive value.

Denoting by $\overline{25}$ the plane passing through the points 2 and 5 and perpendicular to the z-axis, we have moreover

$$16 \mid \overline{25} \mid 347 \quad \ldots \text{G.}$$

We can now show that no two of the faces of our polyhedron intersect each other. Because of the symmetry with respect to the z-axis it suffices to consider pairs of faces formed by a face in the upper line of table 1 and an other which is written *before* it or *under* it in this table. Table 3 gives now for every pair of this type one of the propositions A — G which shows that these two faces cannot intersect each other.

Table 3.

126 — 156 E	346 — 245 B	237 — 126 C	267 — 156 C
235 — 126 G	356 B	156 C	235 C
156 G	124 B	235 D	245 C
245 F	134 B	245 D	356 C
356 — 126 A	467 — 126 B	356 C	124 C
156 A	156 B	124 D	346 C
235 A	235 B	346 C	134 C
245 F	245 B	134 D	467 E
124 F	356 B	67 E	137 D
346 — 126 B	124 B	137 D	237 D
156 B	346 B	457 D	457 D
235 B	134 B	267 — 126 C	157 D
	137 E		

Longer calculation shows that $a = 1$ and $b = 15$ satisfy our above conditions.

We mention finally that the generalized theorem of EULER shows the existence of an infinity of topological types for a polyhedron with the property that every two of its vertices are joined by an edge. It would be of some interest to investigate if all these types can be realized with polyhedra having plane faces and straight edges.

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