doi: 10.14232/actasm-013-255-6

Acta Sci. Math. (*Szeged*)
80 (2014), 603–624

Contractions T for which A is a projection

Carlos S. Kubrusly*

Communicated by L. Kérchy

Abstract. If T is a Hilbert space contraction, then $T^{*n}T^n \xrightarrow{s} A$, where A is a nonnegative contraction. The strong limit A is a projection if and only $T = G \oplus V$, where G is a strongly stable contraction and V is an isometry. This article is an expository paper on the class of contractions T for which A is a projection. After surveying such a class, it is shown that it is quite a large class. Indeed, it includes (i) all contractions whose adjoint has property PF, and also (ii) all contractions whose intersection of the continuous spectrum of its completely nonunitary direct summand with the unit circle has Lebesgue measure zero. Some new questions are investigated as well. For instance, is A a projection for every biquasitriangular contraction T? If so, then every contraction not in class $\mathcal{C}_{00}$ has a nontrivial invariant subspace.

1 Introduction

Throughout this paper $\mathcal{H}$ will stand for a complex Hilbert space. By an operator on $\mathcal{H}$ we mean a bounded linear transformation of $\mathcal{H}$ into itself. Let $\mathcal{N}(T)$ denote the kernel of an operator T (i.e., $\mathcal{N}(T) = T^{-1}(\{0\}) = \{x \in \mathcal{H} \colon Tx = 0\}$), which is a subspace (i.e., a closed linear manifold) of $\mathcal{H}$, and let $\mathcal{R}(T)$ denote the range of T (i.e., $\mathcal{R}(T) = T(\mathcal{H})$), which is a linear manifold of $\mathcal{H}$. A contraction is an operator T such that $\|T\| \le 1$ (i.e., such that $\|Tx\| \le \|x\|$ for every x in $\mathcal{H}$). Let T^* denote the adjoint of T, and let I be the identity operator. An isometry is a contraction V such that $V^*V = I$ (i.e., an operator V such that $\|Vx\| = \|x\|$ for every x in $\mathcal{H}$), and a coisometry is a contraction whose adjoint is an isometry. An operator U is unitary if it is both an isometry and a coisometry (equivalently, if it

Received January 17, 2013, and in final version March 11, 2013.
AMS Subject Classifications: 47A45; 47A15.
Key words and phrases: partially isometric contractions, biquasitriangular operators, invariant subspaces.

*A preliminary draft of this paper was presented at SOTA2 Conference held in Rio in 2001.

is a normal isometry, or a surjective isometry, or still an invertible isometry). If T is a contraction, then $\{T^{*n}T^n\}_{n\geq 0}$ is a bounded monotone sequence of self-adjoint operators (a nonincreasing sequence of nonnegative contractions, actually) so that

$$T^{*n}T^n \xrightarrow{s} A;$$

that is, $\{T^{*n}T^n\}_{n\geq 0}$ converges strongly to an operator A. Basic properties of the strong limit A have been extensively investigated in the current literature (see e.g., [55, p. 38], [8, 12, 33, 34, 36, 41], [28, Chapter 3], and [29, Chapter 6]. In particular, for every contraction T, the strong limit A of $\{T^{*n}T^n\}_{n\geq 0}$ is a nonnegative contraction, which is nonstrict whenever it is not null; that is,

$$O \leq A \leq I \quad \text{ and } \quad \|A\| = 1 \text{ whenever } A \neq O$$

(where O stands for the null operator). Quite recently, a complete characterization of nonnegative contractions A that are strong limits of $\{T^{*n}T^n\}_{n\geq 0}$ was considered in [13]. The above are properties shared by (orthogonal) projections but A is not necessarily a projection (it is not necessarily idempotent).

Example 1. The unilateral weighted shift $T = \text{shift}\{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k\geq 0}$ on ℓ^2_+ is a nonstrict proper contraction for which $A = \text{diag}\{(k+1)(k+2)^{-1}\}_{k\geq 0}$ is a completely nonprojective diagonal (cf. [34] or [28, pp. 51, 52]). In other words, $\|T\| = 1$ and $\|Tx\| < \|x\|$ for every nonzero x in ℓ^2_+ (i.e., T is a nonstrict proper contraction) because the weight sequence $\{w_k\}_{k\geq 0} = \{(k+1)^{1/2}(k+2)^{-1}(k+3)^{1/2}\}_{k\geq 0}$ is increasing in $[\sqrt{3/4}, 1)$ and converges to 1; and $Ax \neq A^2x$ for every nonzero x in ℓ^2_+ (i.e., A is completely nonprojective).

In fact, A *is a projection if and only if it commutes with* T (cf. [8]; also see [34]):

$$A = A^2 \quad \text{if and only if} \quad AT = TA.$$

Since T^* is a contraction whenever T is, the sequence $\{T^nT^{*n}\}_{n\geq 0}$ converges strongly too. Let A_* be its strong limit,

$$T^nT^{*n} \xrightarrow{s} A_*,$$

which, of course, share the same properties of A (by replacing T with T^*).

The present article consists of a research-expository paper on the class of contractions T for which A is a projection. A brief survey on this class is followed by an analysis on the role it plays towards well-known invariant subspace problems. Such a class is fully characterized in Theorem 1 (Section 2) and, in light of this characterization, we call those contractions *asymptotically partially isometric*. Two fundamental results which are enough to unfold many subclasses of it (e.g., cohyponormal, compact, and algebraic contractions) are isolated in Propositions 1 and 2 (Section 3). We link this class with classical open questions on invariant subspaces (Section 4). For example, are biquasitriangular contractions asymptotically partially isometric? If so, then a contraction not in class $\mathcal{C}_{00}$ has a nontrivial invariant subspace, as shown in Theorem 2. The hyponormal counterpart is investigated in Theorem 3.

2 Asymptotically partially isometric contractions

An operator T on a complex Hilbert space $\mathcal{H}$ is weakly, strongly, or uniformly stable (notation: $T^n \xrightarrow{w} O$, $T^n \xrightarrow{s} O$, or $T^n \xrightarrow{u} O$) if the power sequence $\{T^n\}_{n\geq 0}$ converges weakly, strongly, or uniformly to the null operator (i.e., if $\langle T^n x\,;x\rangle \to 0$ for every x in $\mathcal{H}$, $\|T^n x\| \to 0$ for every x in $\mathcal{H}$, or $\|T^n\| \to 0$), respectively. Thus a strongly stable contraction is precisely a contraction of class $\mathcal{C}_{0\cdot}$ and, dually, a contraction whose adjoint is strongly stable is precisely a contraction of class $\mathcal{C}_{\cdot 0}$, so that a contraction T is of class $\mathcal{C}_{00}$ if and only if both T and T^* are strongly stable (see [55, p. 76]). Since

$$\|T^n x\| \to \|A^{\frac{1}{2}} x\| \ \text{ for every } \ x \in \mathcal{H},$$

it follows that a contraction T is strongly stable if and only if $A = O$; that is,

$$T^n \xrightarrow{s} O \quad \text{if and only if} \quad A = O.$$

On the other hand, a contraction T is an isometry if and only if $A = I$; that is,

$$T^*T = I \quad \text{if and only if} \quad A = I,$$

which is readily verified. Actually, for every nonnegative integer n,

$$T^{*n}AT^n = A \quad \text{so that} \quad \|A^{\frac{1}{2}}T^n x\| = \|A^{\frac{1}{2}}x\| \ \text{ for every } \ x \in \mathcal{H}.$$

Thus strongly stable contractions and isometries are classes of contractions T for which A is a trivial projection. Therefore, since *an operator is a backward unilateral shift* (of any multiplicity) *if and only if it is a strongly stable coisometry* — see e.g., [28, p. 88], it follows that a contraction

$$T \text{ is a unilateral shift if and only if } A = I \text{ and } A_* = O.$$

Moreover, although the assertion

$$A = A_* \quad \text{implies} \quad A = A^2 \ \text{ and } \ A_* = A_*^2$$

holds (cf. [34] or [28, p. 53]), the unilateral shift shows that the converse fails.

Let $\mathcal{M}$ be a subspace (i.e., a closed linear manifold) of $\mathcal{H}$. If T is an operator on $\mathcal{H}$, then $T|_{\mathcal{M}}$ is the restriction of T to $\mathcal{M}$. Recall that $\mathcal{H}$ admits the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ is the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. Let V be an isometry on $\mathcal{M}^\perp$. It is clear that the direct (orthogonal) sum $O \oplus V$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ is a partial isometry (a contraction that acts isometrically on the orthogonal complement of its kernel). In fact, this is the simplest nontrivial instance of a power partial isometry (a partial isometry for which all its powers are again partial isometries). It was proved in [19] that *every power partial isometry is a direct sum of a truncated unilateral shift, a unilateral shift, a backward unilateral shift, and a unitary operator* (where, of course, it is understood that not all four direct summands need to be present in every case). Note that the converse holds trivially because each possible direct summand is a power partial isometry. Since truncated shifts are nilpotent, it follows at once that every power partial isometry is a contraction for which $A = A^2$ and $A_* = A_*^2$. (Indeed, $A = O \oplus I \oplus O \oplus I$ and $A_* = O \oplus O \oplus I \oplus I$ if all four direct summands are present.) The above italicized result from [19] can be thought of as a special case of Theorem 1(b) below, where the nilpotent direct summand is extended to a contraction of class $\mathcal{C}_{00}$.

Let T be a contraction on $\mathcal{H}$. If there exists a subspace $\mathcal{M}$ of $\mathcal{H}$ for which

$$T = G \oplus V$$

on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where G is a strongly stable contraction on $\mathcal{M}$, and V is an isometry on $\mathcal{M}^\perp$, then we say that T is an *asymptotically partially isometric contraction*. This means that the power sequence $\{T^n\}_{n\geq 0}$ approaches the sequence of power partial isometries $\{O \oplus V^n\}_{n\geq 0}$ in the strong operator topology;

$$T^n - (O \oplus V^n) \xrightarrow{s} O.$$

The forthcoming Theorem 1(a) says that *a contraction T is asymptotically partially isometric if and only if A is a projection*. We borrow Theorem 1 from [34] (part of it appeared in [8]; also see [28, p. 83]). First recall the von Neumann–Wold decomposition for isometries (e.g., [55, p. 3] or [28, p. 81]): *If T is an isometry on $\mathcal{H}$, then*

$$T = S_+ \oplus U,$$

where $S_+ = T|_{\mathcal{N}(I-A_*)^\perp}$ is a unilateral shift and $U = T|_{\mathcal{N}(I-A_*)}$ is unitary. This can be viewed as a special case of the Nagy–Foiaş–Langer decomposition for contractions [53], [35] (also see [55, p. 8] or [28, p. 76]): *If T is a contraction on $\mathcal{H}$, then*

$$T = C \oplus U,$$

where $C = T|_{\mathcal{U}^\perp}$ *is a completely nonunitary contraction and* $U = T|_{\mathcal{U}}$ *is unitary on* $\mathcal{U} = \mathcal{N}(I-A)\cap\mathcal{N}(I-A_*)$. These decompositions are unique, and a contraction is completely nonunitary (cnu) if it has no nonzero unitary direct summand; equivalently, if the restriction of it to any nonzero reducing subspace is not unitary. The cnu direct summand C is referred to as the cnu part of T, and the unitary U as the unitary part of T. (Contractions of class $\mathcal{C}_{0\cdot}$ or $\mathcal{C}_{\cdot 0}$, and so of class $\mathcal{C}_{00}$, are cnu-unilateral shifts S_+ and backward unilateral shifts S_- are cnu contractions.)

Theorem 1. *Let* T *be a contraction on a Hilbert space. Then*

$$A = A^2 \quad \textit{if and only if} \quad T = G\oplus S_+\oplus U, \tag{(a)}$$

where G *is a* $\mathcal{C}_{0\cdot}$*-contraction (i.e., strongly stable) on* $\mathcal{N}(A)$*,* S_+ *is a unilateral shift on* $\mathcal{N}(I-A)\cap\mathcal{N}(A_*)$*, and* U *is a unitary operator on* $\mathcal{N}(I-A)\cap\mathcal{N}(I-A_*)$*;*

$$A = A^2 \;\textit{ and }\; A_* = A_*^2 \quad \textit{if and only if} \quad T = B\oplus S_-\oplus S_+\oplus U, \tag{(b)}$$

where B *is a* $\mathcal{C}_{00}$*-contraction on* $\mathcal{N}(A)\cap\mathcal{N}(A_*)$ *and* S_- *is a backward unilateral shift (i.e., the adjoint of a unilateral shift) on* $\mathcal{N}(A)\cap\mathcal{N}(I-A_*)$*; and*

$$A = A_* \quad \textit{if and only if} \quad T = B\oplus U. \tag{(c)}$$

Sketchy proof. (a) If $A = A^2$, then $\mathcal{H} = \mathcal{N}(A-A^2)$. However, it can be verified that $\mathcal{N}(A-A^2) = \mathcal{N}(A)\oplus\mathcal{N}(I-A)$, where $\mathcal{N}(A)$ and $\mathcal{N}(I-A)$ are orthogonal, complementary in $\mathcal{N}(A-A^2)$, and T-invariant subspaces. Therefore,

$$T = G\oplus V,$$

where $G = T|_{\mathcal{N}(A)}$ is a strongly stable contraction on $\mathcal{N}(A)$ and $V = T|_{\mathcal{N}(I-A)}$ is an isometry on $\mathcal{N}(I-A)$. Using the von Neumann–Wold decomposition for V,

$$T = G\oplus S_+\oplus U,$$

where $S_+ = V|_{\mathcal{M}}$ is a unilateral shift on $\mathcal{M}$ and $U = V|_{\mathcal{U}}$ is unitary on $\mathcal{U}$, $\mathcal{M}$ and $\mathcal{U}$ being orthogonal complementary subspaces of $\mathcal{N}(I-A)$, so that $G\oplus S_+$ is completely nonunitary. By the Nagy–Foiaş–Langer decomposition it can be shown that

$$\mathcal{U} = \mathcal{N}(I-A)\cap\mathcal{N}(I-A_*) \quad \text{and} \quad \mathcal{M} = \mathcal{N}(I-A)\cap\mathcal{N}(A_*).$$

Conversely, $T = G\oplus S_+\oplus U$ implies $A = O\oplus I\oplus I$.

(b) Since G is a contraction on $\mathcal{N}(A)$, let the operator A'_* on $\mathcal{N}(A)$ be the strong limit of $\{G^nG^{*n}\}$. It can be verified that

$$\mathcal{N}(A)\cap\mathcal{N}(A_*) = \mathcal{N}(A'_*) \quad \text{and} \quad \mathcal{N}(A)\cap\mathcal{N}(I-A_*) = \mathcal{N}(I-A'_*).$$

If, in addition to $A = A^2$, $A_* = A_*^2$, then $A'_* = A'^2_*$, and hence G^* admits a decomposition as in (a), so that $G = B \oplus S_-$, and therefore

$$T = B \oplus S_- \oplus S_+ \oplus U,$$

with $B = G|_{\mathcal{N}(A'_*)} = T|_{\mathcal{N}(A)\cap\mathcal{N}(A_*)}$, a strongly stable contraction on $\mathcal{N}(A)\cap\mathcal{N}(A_*)$ whose adjoint B^* also is strongly stable, and $S_- = G|_{\mathcal{N}(I-A'_*)} = T|_{\mathcal{N}(A)\cap\mathcal{N}(I-A_*)}$ is a strongly stable (so completely nonunitary) contraction on $\mathcal{N}(A)\cap\mathcal{N}(I-A_*)$ whose adjoint is a (completely nonunitary) isometry, so that it is a backward unilateral shift. Conversely, if $T = B \oplus S_- \oplus S_+ \oplus U$, then $A = A^2$ and $A_* = A_*^2$.

(c) If $A = A_*$, then $A = A^2$ and $A_* = A_*^2$ [34], so that T and T^* can be decomposed as in (b). Thus, using the decomposition in (b) for T and T^*, it can be shown that

$$T = B \oplus U,$$

on $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(I - A)$, with $B = T|_{\mathcal{N}(A)}$ and $U = T|_{\mathcal{N}(I-A)}$, where B is a $\mathcal{C}_{00}$-contraction on $\mathcal{N}(A)$ and U is unitary on $\mathcal{N}(I-A)$. Conversely, if $T = B \oplus U$, then $A = A_* = O \oplus I$. (For the detailed proof see [29, p. 60–62].) ■

It is understood that any of the above direct summands may be missing and, if both summands S_- and S_+ are present, they may have distinct (finite or infinite) multiplicities. According to the Nagy–Foiaş–Langer decomposition for a contraction $T = C \oplus U$, Theorem 1(a) says that C is of class $\mathcal{C}_{0\cdot}$ (i.e., C is strongly stable) if and only if $A = A^2$ and the direct summand S_+ is missing in (a), and Theorem 1(c) says that C is of class $\mathcal{C}_{00}$ if and only if $A = A_*$.

3 Two large classes of asymptotically partially isometric contractions

Asymptotically partially isometric contractions are precisely those contractions T for which A is a projection (Theorem 1(a)). Next we isolate two fundamental results (Propositions 1 and 2 below) which ensure that such a class is quite large.

Consider the following definition from [9] (see also [58] and [31]). A contraction T has property PF (a short for Putnam–Fuglede) if either T^* is not intertwined to any isometry or, if T^* is intertwined to some isometry V, then the same transformation that intertwines T^* to V also intertwines T to the coisometry V^*. In other words, let $\mathcal{K}$ be any nonzero complex Hilbert space, and let $X\colon \mathcal{H} \to \mathcal{K}$ be an arbitrary nonzero bounded linear transformation of $\mathcal{H}$ into $\mathcal{K}$. A contraction T on $\mathcal{H}$ has property PF if, whenever the equation $XT^* = VX$ holds for some isometry

V on $\mathcal{K}$, then $XT = V^*X$. Here are two well-known basic facts on contractions with property PF (rather elementary proofs of these results appeared in [31]).

Every isometry has property PF.
If a coisometry has property PF, then it is unitary .

It is worth remarking that, although property PF for contractions as posed above was introduced in [9], the problem of generalizing (in many directions) the classical Fuglede–Putnam Theorem (namely, if a bounded linear transformation intertwines a couple of normal operators, then it also intertwines their adjoints) has been considered by a large number of authors since [52] — for a review on the pertinent literature the reader is referred to [7]. The proof of the next proposition was borrowed from [31]; a different one can be found in [58]. The proposition says that *if a contraction T has property* PF, *then A_* is a projection*; equivalently, *T is asymptotically partially isometric whenever T^* is a contraction with property* PF.

Proposition 1. *If a contraction T has property PF, then $A_* = A_*^2$. Equivalently, if a contraction T is such that T^* has property PF, then $A = A^2$.*

Proof. Take the nonnegative A and an arbitrary integer $n \geq 0$. It can be shown that [28, Section 3.2] there is an isometry V on $\mathcal{R}(A)^-$ such that $A^{\frac{1}{2}}T = VA^{\frac{1}{2}}$. Hence

$$A^{\frac{1}{2}}T^n = V^nA^{\frac{1}{2}}.$$

If T^* has property PF, then $A^{\frac{1}{2}}T^* = V^*A^{\frac{1}{2}}$ so that $A^{\frac{1}{2}}T^{*n} = V^{*n}A^{\frac{1}{2}}$. Thus

$$A^{\frac{1}{2}}V^n = T^nA^{\frac{1}{2}}$$

because $A^{\frac{1}{2}}$ is self-adjoint. But $A = T^{*n}AT^n$ so that

$$A = T^{*n}A^{\frac{1}{2}}A^{\frac{1}{2}}T^n = T^{*n}A^{\frac{1}{2}}V^nA^{\frac{1}{2}} = T^{*n}T^nA^{\frac{1}{2}}A^{\frac{1}{2}},$$

and therefore $A = A^2$ (since $T^{*n}T^n \xrightarrow{s} A$). ■

The class of contractions that have property PF is large. Recall: an operator T is hyponormal, paranormal, or dominant if $O \leq T^*T - TT^*$, $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every x in $\mathcal{H}$, or $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda}I - T^*)$ for every $\lambda \in \mathbb{C}$, respectively, and cohyponormal if its adjoint is hyponormal. These classes are related as follows:

Every hyponormal operator is dominant and paranormal.

Indeed, dominant contractions and paranormal contractions have property PF (see e.g., [9], [58], and the references therein):

If a contraction is dominant or paranormal, then it has property PF.

Thus hyponormal (in particular, normal) contractions have property PF. Therefore, by Proposition 1, dominant and paranormal contractions T are such that $A_* = A_*^2$:

If a contraction T is dominant or paranormal, then $A_* = A_*^2$.
If T is a cohyponormal contraction, then $A = A^2$.

Recalling that T is normal if and only if it is hyponormal and cohyponormal (i.e., if T commutes with T^*), the above implication ensures that $A = A^2$ and $A_* = A_*^2$ if T is normal. Actually, if T is a normal contraction, then $T^{*n}T^n = T^nT^{*n}$ for every integer $n \geq 0$ so that $A = A_*$ trivially, which implies that $A = A^2$ and $A_* = A_*^2$:

If T is a normal contraction, then $A = A_*$, and hence $A = A^2$ and $A_* = A_*^2$.

The preceding observation plus Theorem 1(c) lead to the following result.

If T is a normal contraction, then $T = B \oplus U$, where B is a normal $\mathcal{C}_{00}$-contraction on $\mathcal{N}(A)$ and U is a unitary operator on $\mathcal{N}(I - A)$.

The converse of Proposition 1 fails; even a stronger version fails. For instance, if T is a backward unilateral shift (i.e., $T = S_+^*$ and $T^* = S_+$), then $A = O$ and $A_* = I$ but T does not have property PF (it is a nonunitary coisometry):

A contraction T with $A = A^2$ and $A_* = A_*^2$ may not have property PF.

Consider the Sz. Nagy–Foiaş–Langer decomposition $T = C \oplus U$ of a contraction T. In fact, it was proved in [9] that *a contraction T has property PF if and only if its completely nonunitary direct summand is of class $\mathcal{C}_{\cdot 0}$*. That is,

$T = C \oplus U$ is a contraction with property PF if and only if C is of class $\mathcal{C}_{\cdot 0}$

(see also [31]). Thus (Theorem 1(c)) contractions T and T^* have property PF if and only if their completely nonunitary direct summands are of class $\mathcal{C}_{00}$; that is,

T and T^* have property PF if and only if $A = A_*$.

Perhaps a systematic investigation on asymptotically partially isometric contractions has been initiated after Putnam's paper [44]. It contains the first proof that a completely nonunitary cohyponormal contraction is strongly stable and, consequently, that if T^* is a hyponormal contraction, then $T = G \oplus U$, where G is a strongly stable contraction and U is unitary, so that $A = A^2$. Simplified different proofs followed in [38] (see also [57, pp. 113–116]) and in [33] (see also [28, pp. 77–79]) by using a reverse approach. They first verified that $A = A^2$ if T is a cohyponormal contraction and then concluded that a completely nonunitary cohyponormal contraction is strongly stable (thus stressing the role played by contractions for which A is a projection). This was extended to paranormal contractions in [38], and to dominant contractions in [14] and [56], which are classes of contractions that include the hyponormal contractions. Summing up:

If a contraction $T = C \oplus U$ is dominant or paranormal, then $C \in \mathcal{C}_{\cdot 0}$.
If $T = C \oplus U$ is a cohyponormal contraction, then $C \in \mathcal{C}_{0\cdot}$.

Extentions to k-paranormal contractions (which include the paranormal) and to k-quasihyponormal contractions (which include the hyponormal) have been discussed in the literature. An operator T is k-paranormal if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\,\|x\|^k$ for some integer $k \geq 1$ and every $x \in \mathcal{H}$ (a paranormal is simply a 1-paranormal operator), and an operator T is k-quasihyponormal if $O \leq T^{*k}(T^*T - TT^*)T^k$ for some integer $k \geq 0$ (a hyponormal is simply a 0-quasihyponormal operator — if $k = 1$, then T is called quasihyponormal). The following result is from [11].

If a contraction $T = C \oplus U$ is k-paranormal or k-quasihyponormal, then $C \in \mathcal{C}_{\cdot 0}$.

More extensions along these lines (i.e., for classes of contractions that include the hyponormal) can be found in [10] and [39]. Extensions along different lines have also been discussed in the literature. For instance, extension to bicontractions (i.e., to a pair of commuting contractions) was considered in [26], and extension to A'-contractions in [49, 50]. That is, extension to operators T for which there is a positive A' such that $T^*A'T \leq A'$. If the equality holds, then T is said to be an A'-isometry. (In particular, if T is a contraction, then it is an A-isometry and also an I-contraction). Further extensions to noncontractions have been investigated in [51] by considering the asymptotic limit of T which generalizes the strong limit A, as defined in [21, 22] for power bounded operators T, or in [23, 24] for operators whose power sequence satisfies some regularity condition weaker than power boundedness.

Another approach to asymptotically partially isometric contractions evolving in a different direction and including classes of contractions not related to the above examples will be proved next. Let $\sigma(T)$ denote the spectrum of an operator T and consider its classical partition $\sigma(T) = \sigma_P(T) \cup \sigma_R(T) \cup \sigma_C(T)$, where $\sigma_P(T)$ is the point spectrum (i.e., the set of all eigenvalues of T), $\sigma_R(T) = \sigma_P(T^*)^* \backslash \sigma_P(T)$ is the residual spectrum, and $\sigma_C(T) = \sigma(T) \backslash (\sigma_P(T) \cup \sigma_R(T))$ is the continuous spectrum. Let μ denote the Lebesgue measure on the unit circle $\partial\mathbb{D}$ (where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$). Consider again the Nagy–Foiaş–Langer decomposition.

Proposition 2. *If a contraction $T = C \oplus U$ is such that $\mu(\sigma_C(C) \cap \partial\mathbb{D}) = 0$, then $A = A_*$. Equivalently, $\mu(\sigma_C(C) \cap \partial\mathbb{D}) = 0$ implies that T and T^* have property PF, which in turn implies that $A = A^2$ and $A_* = A_*^2$.*

Proof. Let $T = C \oplus U$ be a contraction, where C is a completely nonunitary contraction and U is unitary (as always, any of the above direct summands may be missing). If C is missing then $T = U$, and $A = A_*$ trivially, since U is normal. Thus suppose C is not missing. Recall: *every completely nonunitary contraction is weakly stable* [15, p. 55], and *a weakly stable contraction C is such that $\sigma_P(C) \cup \sigma_R(C)$ is included in the open unit disc* [29, p. 80]. That is (also see [28, pp. 106,114]),

$$C \text{ is cnu implies } C^n \xrightarrow{w} O \text{ which implies } \sigma_P(C) \cup \sigma_R(C) \subseteq \mathbb{D}.$$

Hence

$$\mu(\sigma_C(C) \cap \partial\mathbb{D}) = 0 \quad \text{implies} \quad \mu(\sigma(C) \cap \partial\mathbb{D}) = 0.$$

Since C is a cnu contraction, it follows that

$$\mu(\sigma(C) \cap \partial\mathbb{D}) = 0 \quad \text{implies} \quad C \in \mathcal{C}_{00}$$

(see [55, p. 90], which originated from [54, p. 127]). This means that C and C^* are strongly stable (i.e., $T = B \oplus U$), and therefore $A = A_* = O \oplus I$ according to Theorem 1(c). The rest of the statement follows at once by Proposition 1. ■

If T is a compact contraction, then $A = A_*$, and hence $A = A^2$ and $A_* = A_*^2$.

Indeed, compact (countable spectrum) and algebraic (finite spectrum) contractions are asymptotically partially isometric by Proposition 2 (and T is compact or algebraic if and only if T^* is). Quasinilpotent (one-point spectrum) contractions are also included, but these are trivially asymptotically partially isometric; they lie in $\mathcal{C}_{00}$. Another particular case of Proposition 2 reads as follows.

$$\text{If } T = C \oplus U \text{ and } \sigma_C(C) \cap \partial\mathbb{D} = \varnothing, \text{ then } C \in \mathcal{C}_{00}, \text{ and so } A = A_*.$$

Such a particular case can be readily verified without the help of the measure theoretical result from [55, p. 90]. Actually, the previous argument ensures that if $\sigma_C(C) \cap \partial\mathbb{D} = \varnothing$, then $\sigma(C) \subset \mathbb{D}$, and so $r(C) < 1$, which means that $C^n \xrightarrow{u} O$. ($r(\,\cdot\,)$ denotes spectral radius; for further equivalent conditions to uniform stability see, e.g., [28, p. 11].) Hence $C^n \xrightarrow{s} O$ and $C^{*n} \xrightarrow{s} O$; that is, $C \in \mathcal{C}_{00}$.

4 Biquasitriangular contractions

Are they asymptotically partially isometric? Before defining biquasitriangular operators and considering this question we need a finer analysis of the spectrum. Let

$\mathcal{B}[\mathcal{H}]$ denotes the algebra of all operators on $\mathcal{H}$. Set $\mathcal{F}_\ell = \{T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty\}$, $\mathcal{F}_r = \{T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T^*) < \infty\}$, $\mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$, and $\mathcal{W} = \{T \in \mathcal{F} \colon \dim \mathcal{N}(T) = \dim \mathcal{N}(T^*)\}$. These are the classes of left semi-Fredholm, right semi-Fredholm, Fredholm, and Weyl operators, respectively. Let $\sigma_{\ell e}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_\ell\}$, $\sigma_{re}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_r\}$, $\sigma_e(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}\}$, and $\sigma_w(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{W}\}$ be the left essential spectrum, the right essential spectrum, the essential spectrum, and the Weyl spectrum of an arbitrary operator $T \in \mathcal{B}[\mathcal{H}]$, respectively. Recall that $\sigma_{\ell e}(T) \cup \sigma_{re}(T) = \sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$, that $\sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_e(T) = \sigma_w(T)$ if and only if $\sigma_e(T)$ has no holes (of nonzero index) and no pseudoholes, and set $\sigma_0(T) = \sigma(T) \backslash \sigma_w(T)$ (see, e.g., [30, pp. 131–162]).

From now on let $\mathcal{H}$ be a complex infinite-dimensional *separable* Hilbert space. An operator $T \in \mathcal{B}[\mathcal{H}]$ is quasitriangular (or quasidiagonal) if there is a sequence $\{P_n\}$ of finite-rank projections in $\mathcal{B}[\mathcal{H}]$ that converges strongly to the identity operator and $\{(I - P_n)TP_n\}$ (or $\{TP_n - P_nT\}$) converges uniformly to the null operator [16]. It is plain that T is quasidiagonal if and only if T^* is (since orthogonal projections are self-adjoint). An operator T is biquasitriangular if both T and T^* are quasitriangular. For a collection of results on quasitriangular and biquasitriangular operators see, for instance, [40, pp. 25–30] and [20, pp. 163–192]. In particular [40, p. 37],

$$T \in \mathcal{B}[\mathcal{H}] \text{ is biquasitriangular if and only if } \sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_e(T) = \sigma_w(T).$$

Since every operator on $\mathcal{H}$ with a countable spectrum is quasitriangular (see, e.g., [40, p,29]), it follows that the samples of contractions for Proposition 2 (compact, algebraic and quasinilpotent) are all biquasitriangular (since adjoint of compact, algebraic and quasinilpotent are again compact, algebraic and quasinilpotent).

'One of the most important, most difficult, and most exasperating unsolved problems of operator theory is the problem of invariant subspaces. The question is simple to state: does every operator on an infinite-dimensional [*separable, complex*] *Hilbert space have a non-trivial invariant subspace? "Non-trivial" means different from both* 0 *and the whole space, "invariant" means that the operator maps it to itself'* [18, p. 100]. An operator that has a nontrivial invariant subspace is called intransitive, otherwise it is called transitive. Let "nis" mean "nontrivial invariant subspace".

The Riesz Decomposition Theorem (see, e.g., [45, p. 32]) ensures that if the spectrum $\sigma(T)$ of an operator T is disconnected then it has a nis:

If there is an operator $T \in \mathcal{B}[\mathcal{H}]$ without a nis, then $\sigma(T)$ is connected.

(In this case, the nonempty compact $\sigma(T)$ has no isolated point — a bounded perfect set.) Recall that $\mathcal{N}(\lambda I - T)$ and $\mathcal{R}(\lambda I - T)^-$ are T-invariant subspaces for every λ in $\mathbb{C}$. Consider the classical partition of the spectrum. Since $\{0\} \neq \mathcal{N}(\lambda I - T)$ for every $\lambda \in \sigma_P(T)$, and since T has a nis if and only if T^* has a nis, we may infer:

If there is an operator $T \in \mathcal{B}[\mathcal{H}]$ without a nis, then $\sigma(T) = \sigma_C(T)$.

Also recall that $\sigma_C(T) \subseteq \sigma_{\ell e}(T) \cap \sigma_{re}(T)$ (see e.g., [30, p. 146]). Thus $\sigma(T) = \sigma_C(T)$ implies $\sigma(T) = \sigma_{\ell e}(T) \cap \sigma_{re}(T)$. Note that, if $\lambda \notin \sigma_{\ell e}(T)$ (i.e., if $(\lambda I - T) \in \mathcal{F}_\ell$), then $\mathcal{R}(\lambda I - T)$ is closed and $\dim \mathcal{N}(\lambda I - T) < \infty$. If $\mathcal{R}(\lambda I - T) \neq \mathcal{H}$, then it is a nis for T. If $\mathcal{N}(\lambda I - T) \neq \{0\}$, then it is a nis for T (λ is an eigenvalue of T). If $\mathcal{R}(\lambda I - T) = \mathcal{H}$ and $\mathcal{N}(\lambda I - T) = \{0\}$, then $\lambda I - T$ is invertible, which means that $\lambda \notin \sigma(T)$. Outcome: if $\lambda \in \sigma(T) \backslash \sigma_{\ell e}(T)$, then T has a nis. Since $\mathcal{M}$ is a nis for T if and only if $\mathcal{M}^\perp$ is nis for T^*, it also follows that if $\lambda \in \sigma(T) \backslash \sigma_{re}(T)$, then T has a nis. Thus, since $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T)$, we may claim:

If there exists $T \in \mathcal{B}[\mathcal{H}]$ without a nis, then $\sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_e(T) = \sigma(T)$.

The previous spectral equivalent definition of biquasitriangular operators ensures that, if there is a $T \in \mathcal{B}[\mathcal{H}]$ with no nis, then it is biquasitriangular (since $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T)$), and $\sigma_0(T) = \sigma(T) \backslash \sigma_w(T) = \varnothing$. In fact, recalling that $\sigma_0(T)$ consists of eigenvalues only (see, e.g., [30, p. 151]), it is already empty if T has no nis. Hence we get the following result.

If there exists $T \in \mathcal{B}[\mathcal{H}]$ with no nis, then it is biquasitriangular.

Equivalently, *if $T \in \mathcal{B}[\mathcal{H}]$ is not biquasitriangular, then it has a nis.* The closure (in $\mathcal{B}[\mathcal{H}]$) of the set of all nilpotent operators coincides with the set of all biquasitriangular operators T for which $\sigma_e(T)$ and $\sigma(T)$ are both connected and $0 \in \sigma_e(T)$ (see, e.g., [40, p. 40]). If T has no nis, then it is biquasitriangular, $\sigma_e(T) = \sigma_w(T) = \sigma(T)$ is connected, and either $0 \in \sigma_w(T)$ or $0 \notin \sigma(T)$ (since $\sigma_0(T) \subseteq \sigma_P(T)$ and, if T has no nis, then $\sigma_P(T) = \varnothing$). However, by replacing T with $\lambda I - T$ if necessary (which share the same lattice of invariant subspaces, and are such that $\sigma(\lambda I - T) = \lambda - \sigma(T)$), it follows that there is no loss of generality in assuming that $0 \in \sigma(T)$, so that $0 \in \sigma_w(T)$. Thus the previous implication can be tightened as follows.

If there exists $T \in \mathcal{B}[\mathcal{H}]$ with no nis, then there is a translation of it $\lambda I - T \in \mathcal{B}[\mathcal{H}]$ with no nis that lies in the closure of the nilpotent.

Note that the set of nilpotent operators is trivially included in the set of quasinilpotent operators, which is included in the closure of the set of the nilpotent operators (see, e.g., [40, 40]), which in turn is included in the set of all biquasitriangular operators (as we saw above). It is also worth noticing that operators of the form $N+K$, where N is normal and K compact, are quasitriangular [16], and so they are biquasitriangular (since the adjoint of normal or compact is again normal or compact). In fact, the set of all operators of the form $N+K$ coincides with the set of biquasitriangular that are essentially normal (see e.g., [40, p. 38]). Recall that an operator is essentially normal if its image under the natural quotient map of $\mathcal{B}[\mathcal{H}]$ into the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is normal, where $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is the quotient algebra of $\mathcal{B}[\mathcal{H}]$ modulo the ideal $\mathcal{B}_\infty[\mathcal{H}]$ of all compact operators from $\mathcal{B}[\mathcal{H}]$. Equivalently, an operator T is essentially normal if its self-commutator $D=T^*T-TT^*$ is compact.

The invariant subspace problem remains unsolved for hyponormal operators. In this case, besides the above conditions, there is also the following one [40, p. 50]:

If there is a hyponormal $T\in\mathcal{B}[\mathcal{H}]$ with no nis, then it is of the form $N+K$.

In addition, if a hyponormal T has no nis, then $\operatorname{area}(\sigma(T))>0$ ("area" means planar Lebesgue measure) by the Putnam inequality [42] (also see [6, p. 156] and [37, p. 31]), and $\sigma(T^*T)$ is an interval [43]. Moreover, a deep result from [4] says that $\sigma(T)^\circ=\varnothing$: the spectrum of a hyponormal operator with no nis has empty interior.

Let $\mathcal{N}il$, $\mathcal{A}lg$ and $\mathcal{QN}il$ denote the classes of nilpotent ($T^n=O$ for some positive integer n), algebraic ($p(T)=O$ for some nonzero polynomial p), and quasinilpotent ($\sigma(T)=\{0\}$) operators. Let $\mathcal{QD}$, $\mathcal{BQT}$, and $\mathcal{EN}$ stand for quasidiagonal, biquasitriangular, and essentially normal; and let $\mathcal{N}+\mathcal{K}$ be the class of all operators which are the sum of normal plus compact (including trivially the normal and compact operators individually). Recall that $\mathcal{B}_\infty[\mathcal{H}]$, $\mathcal{N}+\mathcal{K}$, $\mathcal{QD}$, $\mathcal{EN}$, and $\mathcal{BQT}$ are closed in $\mathcal{B}[\mathcal{H}]$ and invariant under compact perturbation (see e.g., [40, p. 38, 40] and [20, p. 170, 172]). These classes are related as follows (see e.g., [40, p. 37–40, 48]).

$$\mathcal{N}+\mathcal{K}=\mathcal{QD}\cap\mathcal{EN}=\mathcal{BQT}\cap\mathcal{EN}\subset\mathcal{QD}\subset\mathcal{BQT},$$

$$\mathcal{N}il\subset\mathcal{A}lg\subset\mathcal{A}lg^-=\mathcal{BQT},\quad \mathcal{N}il\subset\mathcal{QN}il\subset\mathcal{N}il^-\subset\mathcal{BQT},\quad \mathcal{QN}il\cap\mathcal{EN}\subset\mathcal{B}_\infty[\mathcal{H}].$$

Further equivalent expressions for $\mathcal{BQT}$ go as follows [20, p. 171]:

$$\begin{aligned}
\mathcal{BQT} &= \{T\in\mathcal{B}[\mathcal{H}]\colon T \text{ is similar to a normal operator with finite spectrum}\}^-\\
&= \{T\in\mathcal{B}[\mathcal{H}]\colon T \text{ is similar to a normal operator}\}^-\\
&= \{T\in\mathcal{B}[\mathcal{H}]\colon T \text{ is similar to a quasidiagonal operator}\}^-\\
&= \{T\in\mathcal{B}[\mathcal{H}]\colon \sigma(T) \text{ is totally disconnected}\}^-\\
&= \{T\in\mathcal{B}[\mathcal{H}]\colon \sigma(T) \text{ has empty interior}\}^-.
\end{aligned}$$

We now return to contractions. Note that the invariant subspace problem is invariant under scalar multiplication (T and αT have the same lattice of invariant subspaces for every $\alpha \in \mathbb{C}$) — there exists an operator without a nis if and only if there exists a contraction without a nis.

If a contraction has no nis, then it is completely nonunitary,

which is a trivial corollary of Nagy–Foiaş–Langer decomposition for contractions. On the other hand, another deep result from [5] (also see [1]) gives an important condition. *A contraction whose spectrum includes the unit circle has a nis*:

If a contraction T has no nis, then it $\partial\mathbb{D} \not\subseteq \sigma(T)$.

Recall that a hyperinvariant subspace for an operator is a subspace that is invariant for every operator that commutes with it, which is a particular case of invariant subspace. A nonscalar contraction without a nontrivial hyperinvariant subspace is either of class $\mathcal{C}_{00}$, or of class $\mathcal{C}_{01}$, or of class $\mathcal{C}_{10}$ [27]. We shall be interested in the following especial case [28, p. 85].

If a contraction has no nis, then it is of class $\mathcal{C}_{00}$, or $\mathcal{C}_{10}$, or $\mathcal{C}_{01}$

and, if it is of class $\mathcal{C}_{10}$ or of class $\mathcal{C}_{01}$, then A or A_* is a proper contraction, respectively (i.e., either $\|Ax\| < \|x\|$ or $\|A_*x\| < \|x\|$ for $x \neq 0$). Since a completely nonunitary contraction has property PF if and only if it is of class $\mathcal{C}_{\cdot 0}$ [9] (and since a contraction has property PF if and only if its completely nonunitary direct summand has property PF), we get the next result [31].

If neither T nor T^* have property PF, then the contraction T has a nis.

Question 1. Are biquasitriangular contractions asymptotically partially isometric?

Biquasitriangular means that the operator and its adjoint are quasitriangular, Thus we can rewrite the above question as follows. Is it true that if

T and T^* are quasitriangular contractions, then $A = A^2$ and $A_* = A_*^2$?

This question can be tightened as follows.

Is it true that if T is a quasitriangular contractions, then $A = A^2$?

If Question 1 has an affirmative answer, then a biquasitriangular contraction T admits a decomposition $T = B \oplus S_- \oplus S_+ \oplus U$ (Theorem 1(b)) but now just some direct summands might be missing; a unilateral shift S_+ is not quasitriangular [16], although the direct sum $S_- \oplus S_+$ may be — that is, if one is the adjoint of the other (same multiplicity), then their direct sum is quasidiagonal [17]. Therefore, it is tempting to think that Question 1 might be tightened as follows.

Question 1′. Is it true that if T is a biquasitriangular contraction, then $A = A_*$?

Consider once again the Nagy–Foiaş–Langer decomposition $T = C \oplus U$ for a contraction T, where C is completely nonunitary and U is unitary. According to Theorem 1(c), Question 1′ can be rewritten in terms of C as follows.

Is the completely nonunitary part of a biquasitriangular contraction of class $\mathcal{C}_{00}$?

Equivalently, is it true that if

T is a biquasitriangular contraction, then T and T^* have property PF ?

An affirmative answer to Question 1′ would imply that $T = B \oplus U$ (Theorem 1(c)), which trivially implies an affirmative answer to Question 1. Recalling that U is biquasitriangular (it is normal), and that a (countable) direct sum of biquasitriangular operators is again biquasitriangular [16], the situation here is simpler; any direct summand might be missing.

Answer 1′. No. $T = S_+ \oplus S_+^*$ is a biquasitriangular contraction for which $A \neq A_*$. Indeed, if S_+ is a unilateral shift (of multiplicity one), then $S_+ \oplus S_+^*$ is quasitriangular [17]. Since it is unitarily equivalent to its own adjoint, it follows that it is biquasitriangular. Hence $S_+ \oplus S_+^*$ is a completely nonunitary biquasitriangular contraction which, of course, is not of class $\mathcal{C}_{00}$. In fact, if $T = S_+ \oplus S_+^*$, then $A = I \oplus O$ and $A_* = O \oplus I$. Thus the contraction $S_+ \oplus S_+^*$ supplies a negative answer to Question 1′, but not to Question 1; $S_+ \oplus S_+^*$ is an asymptotically partially isometric biquasitriangular contraction. ■

Example 2. Let T be the unilateral weighted shift of Example 1. It is a hyponormal (its positive weight sequence is increasing) contraction. Since T is not asymptotically partially isometric, we should verify whether it survives Question 1. Yes, it does; it is not quasitriangular (reason: $T^*T = \operatorname{diag}\{w_k^2\}_{k\geq 0} \geq (3/4)I$ and $\mathcal{N}(T^*) \neq \{0\}$ [17, p. 904]); neither is $O \oplus T$ [16, p. 293]. Note that $T \notin \mathcal{N} + \mathcal{K} = \mathcal{QD} \cap \mathcal{EN} = \mathcal{BQT} \cap \mathcal{EN}$ since $T \notin \mathcal{BQT}$, but $T \in \mathcal{EN}$ (i.e., $T^*T - TT^* \in \mathcal{B}_\infty[\mathcal{H}]$) and so $T \notin \mathcal{QD}$.

A question simpler than Question 1 (in the sense that an affirmative answer to Question 1 would trivially imply an affirmative answer to it) deals with contractions in the closure $\mathcal{N}il^-$ of the nilpotent operators.

Question 2. Are contractions in $\mathcal{N}il^{-}$ asymptotically partially isometric?

Questions 1 and 2 have at least one important consequence: an affirmative answer to Question 1 leads to an affirmative answer to Question 2, which leads to an affirmative answer to a classical open question (for equivalent versions, see [27]).

Question 3. Does a contraction not in $\mathcal{C}_{00}$ have a nis?

Theorem 2. *If every biquasitriangular contraction* (*or if every contraction in* $\mathcal{N}il^{-}$) *is asymptotically partially isometric, then every contraction not in* $\mathcal{C}_{00}$ *has a nontrivial invariant subspace.*

Proof. Suppose there is a contraction without a nontrivial invariant subspace. Equivalently, suppose there is a contraction T with no nis in $\mathcal{N}il^{-} \subset \mathcal{BQT}$. If every contraction in $\mathcal{BQT}$ (in particular, in $\mathcal{N}il^{-}$) is asymptotically partially isometric, then T is such that $A = A^2$ and $A_* = A_*^2$. However, if a contraction T with $A = A^2$ and $A_* = A_*^2$ has no nis, then $T \in \mathcal{C}_{00}$ (i.e., $A = A_* = O$) by Theorem 1(b). (Indeed, $T = B \in \mathcal{C}_{00}$ since the other possible direct summands S_-, S_+ and U — isometries and coisometries — have a nis.) Equivalently (under the above assumption), if $T \notin \mathcal{C}_{00}$, then it has a nis. ∎

We saw that if a hyponormal operator has no nis, then it is of the form $N + K$. There is a myriad of attributes that a hyponormal contraction without a nis (if there exists such a contraction) must satisfy. Among them are the following.

If a hyponormal contraction has no nis, then it is of class $\mathcal{C}_{00}$ or of class $\mathcal{C}_{10}$

and, if it is of class $\mathcal{C}_{10}$, then the nonnegative A is a completely nonprojective (i.e., $Ax \neq A^2x$ for $x \neq 0$) nonstrict proper contraction (i.e., $\|A\| = 1$ and $\|Ax\| < \|x\|$ for $x \neq 0$) [32]. Besides, it can be inferred from [24] that A has no eigenvalue, and hence $\sigma(A) = \sigma_C(A)$. (Reason: since $T^*AT = A$, so that A is a T-Toeplitz operator, if A has an eigenvalue, which must be positive because A is nonnegative and completely nonprojective, then the restriction of T to a nonzero invariant subspace is similar to an isometry, and so T has a nontrivial invariant subspace — cf. [24, Theorem 14, p. 394] or [25, Theorem 6, p. 126]). Still in this case ($T \in \mathcal{C}_{10}$), Proposition 2 ensures that $\mu\big(\sigma(T) \cap \partial\mathbb{D}\big) > 0$ (where $\sigma(T) = \sigma_C(T)$ is connected, and μ is the Lebesgue measure on $\partial\mathbb{D}$). In any case ($T \in \mathcal{C}_{00} \cup \mathcal{C}_{10}$), if a hyponormal contraction T has no nis, then the cnu T is a nonstrict proper contraction (i.e., $\|T\| = 1$ and $\|Tx\| < \|x\|$ for $x \neq 0$) and its self-commutator $O \leq D = T^*T - TT^*$ is a strict contraction (i.e., $\|D\| < 1$) [32]. Moreover, D is trace-class (thus compact) with $\|D\|_1 \leq 1$ (where $\|\cdot\|_1$ denotes trace-norm) by the Berger–Shaw Theorem [2,3] (also see [6, p. 152] and [37, p. 127]). As we had already seen, a hyponormal contraction with no nis is essentially normal, which means that D is compact; and it is a strict contraction even if it is a rank-one operator.

Again, questions simpler than Question 1 (in the sense that an affirmative answer to Question 1 would trivially imply affirmative answers to them) read as follows.

Question 4. Are quasidiagonal contractions asymptotically partially isometric?

In other words, since T is quasidiagonal if and only if T^* is, is it true that if

$$T \text{ is a quasidiagonal contraction, then } A = A^2 \text{ and } A_* = A_*^2 \text{ ?}$$

In particular, we get the following even simpler (in the same sense) question.

Question 5. Is a contraction of the form $N + K$ asymptotically partially isometric?

Equivalently, is it true that if

$$T \text{ is a contraction of the form } N + K, \text{ then } A = A^2 \text{ and } A_* = A_*^2 \text{ ?}$$

Observe that, since $\mathcal{N}+\mathcal{K} \subset \mathcal{QD} \subset \mathcal{BQT}$, an affirmative answer to Question 1 implies an affirmative answer to Question 4, which in turn implies an affirmative answer to Question 5. Thus Questions 4 and 5 also have at least one important consequence, namely, an affirmative answer to Question 4 leads to an affirmative answer to Question 5, which in turn leads to an affirmative answer to another classical open question — the hyponormal version of Question 3:

Question 6. Does a hyponormal contraction not in $\mathcal{C}_{00}$ have a nis?

Theorem 3. *If every quasidiagonal contraction* (*or if every contraction of the form* $N + K$) *is asymptotically partially isometric, then every hyponormal contraction not in* $\mathcal{C}_{00}$ *has a nontrivial invariant subspace.*

Proof. Let T be a hyponormal contraction. Suppose T has no nis. Recall that if there is an hyponormal operator without a nis, then it lies in $\mathcal{N}+\mathcal{K} \subset \mathcal{QD}$. Thus $T \in \mathcal{N}+\mathcal{K} \subset \mathcal{QD}$. Suppose every contraction in $\mathcal{QD}$ (in particular, in $\mathcal{N}+\mathcal{K}$) is asymptotically partially isometric. Then T is such that $A = A^2$. However, as we saw before, if a hyponormal contraction T with no nis is such that $A = A^2$ then $T \in \mathcal{C}_{00}$ (because if it is not in $\mathcal{C}_{00}$, then it must be in $\mathcal{C}_{10}$ with a completely nonprojective A.) Equivalently (under the above assumption), if $T \notin \mathcal{C}_{00}$, then it has a nis. ∎

An immediate corollary of Theorem 3 with a sharper statement reads as follows.

Corollary 1. *If every quasidiagonal hyponormal contraction* (*or if every hyponormal contraction of the form* $N + K$) *is asymptotically partially isometric, then every hyponormal contraction not in* $\mathcal{C}_{00}$ *has a nontrivial invariant subspace.*

5 Final remarks

Observe through Theorem 1 that, if A is a projection, then the contraction T has a nis, provided that $A \neq O$ — that is, provided that $T \notin \mathcal{C}_{0\cdot}$ (in fact, a nonscalar contraction $T \notin \mathcal{C}_{01} \cup \mathcal{C}_{10} \cup \mathcal{C}_{00}$ has a nontrivial hyperinvariant subspace). Therefore, the invariant subspace problem is naturally linked to asymptotically partially isometric contractions (witness: Question 3, which has been exhaustively investigated in the literature, is naturally linked to Theorem 1(c)). However, the invariant subspace problem is not the central theme of this expository paper (for expository papers on the invariant subspace problem the reader is referred to, for instance, [1, 46, 47]). Actually, the present paper is not built around the above contact point with the invariant subspace problem, in the hope that one could solve the invariant subspace problem for many operators by showing that they fit in the above scheme. On the contrary, the central focus of the paper is asymptotically partially isometric contraction "per se", which has shown to be a large class of operators, being relevant even for investigating invariant subspace problems.

Perhaps another question that might arise when investigating contractions T for which A is a projection would be related to the shape of the spectrum of T. For instance, *is there any property that the shape of the spectrum of a contraction* T *must possess if* A *is a projection*? The answer is "no". Indeed, since normal contractions are such that A is a projection, it follows that every compact subset Ω of the complex plane $\mathbb{C}$, included in the closed unit disc $\mathbb{D}^-$, is the spectrum of a contraction T for which A is a projection. Example: let $\Omega \subseteq \mathbb{D}^- \subset \mathbb{C}$ be any compact set included in $\mathbb{D}^-$, which is separable — either because it is a compact set on metric space, or because it is subset of a separable metric space — and hence there is a countable set $\Lambda \subseteq \Omega$ dense in Ω. Thus take a diagonal operator T on ℓ^2_+ whose diagonal entries $\{\lambda_k\}$ consist of an enumeration of the elements of Λ. Therefore, the shape of the spectrum $\sigma(T) = \Lambda^- = \Omega$ of T does not affect the property of A being a projection.

Acknowledgement. It is my pleasure to thank the referee who did a careful reading of the manuscript, raising many sensible remarks.

Note added in proof

Part of this paper was presented at the Sz.-Nagy Centennial Conference held in Szeged, Hungary (June 2013). Vladimir Müller gave a negative answer to Question 1: the bilateral weighted shift $T = \text{shift}\{w_k\}$ on ℓ^2 with $w_k = \frac{-k}{1-k}$ for $k < 0$ and $w_k = \frac{(k+1)(k+4)}{(k+2)(k+3)}$ for $k \geq 0$ is a $\mathcal{BQT}$ contraction for which $A \neq A^2$. Indeed, this is an injective ($w_k > 0$) proper $\mathcal{C}_{10}$-contraction ($w_k < 1$) such that $\lim_{k\to-\infty} w_k = \lim_{k\to\infty} w_k = 1$, and so $\sigma_P(T) = \sigma_P(T^*) = \varnothing$ ([48, Proposition 15, p. 72, and Theorem 9, p. 71]), which implies that it is $\mathcal{BQT}$. Since T is a proper $\mathcal{C}_{10}$-contraction, A is completely nonprojective ($0 < \|Ax\| = \|A^{\frac{1}{2}}TA^{\frac{1}{2}}x\| < \|A^{\frac{1}{2}}x\|$ for $x \neq 0$). György Gehér gave a negative answer to Question 5 by pointing out that a perturbation of a bilateral (unweighted) shift U by a compact bilateral weighted shift K, say, $T = U - K$ for $K = \text{shift}\{w_k\}$ on ℓ^2 with $\lim_{|k|\to\infty} w_k = 0$ lies in $\mathcal{N} + \mathcal{K}$. Selecting the nonnegative weights w_k such that T is a contraction ($0 \leq 1 - w_k \leq 1$) and $A \neq A^2$, a negative answer to Question 5 (and so to Questions 4 and 1) is supplied.

References

[1] H. Bercovici, Notes on invariant subspaces, *Bull. Amer. Math. Soc.*, **23** (1990), 1–36.

[2] C. A. Berger and B. I. Shaw, Self-commutators of multicyclic hyponormal operators are always trace class, *Bull. Amer. Math. Soc.*, **79** (1973), 1193–1199.

[3] C. A. Berger and B. I. Shaw, Intertwining, analytic structures, and the trace norm estimate, *Proceedings of a Conference on Operator Theory, Halifax*, 1973, Lecture Notes in Math. Vol. 345, Springer, Berlin, 1973, 1–6.

[4] S. W. Brown, Hyponormal operators with thick spectra have invariant subspaces, *Ann. Math.*, **125** (1987), 93–103.

[5] S. W. Brown, B. Chevreau and C. Pearcy, On the structure of contractions operators II, *J. Functional Anal.*, **76** (1988), 30–55.

[6] J. B. Conway, *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs Vol. 36, Amer. Math. Soc., Providence, 1991.

[7] B. P. Duggal, On generalised Putnam-Fuglede theorems, *Monatsh. Math.*, **107** (1989), 309–332.

[8] B. P. Duggal, On unitary parts of contractions, *Indian J. Pure Appl. Math.*, **25** (1994), 1243–1247.

[9] B. P. Duggal, On characterising contractions with $\mathcal{C}_{10}$ pure part, *Integral Equations Operator Theory*, **27** (1997), 314–323.

[10] B. P. Duggal, S. V. Djordjević and C. S. Kubrusly, Hereditarily normaloid contractions, *Acta Sci. Math. (Szeged)*, **71** (2005), 337–352.

[11] B. P. Duggal and C. S. Kubrusly, Paranormal contractions have property PF, *Far East J. of Math. Sci.*, **14** (2004), 237–249.

[12] E. Durszt, Contractions as restricted shifts, *Acta Sci. Math.* (*Szeged*), **48** (1985), 129–134.

[13] G. P. Gehér, Positive operators arising asymptotically from contractions, *Acta Sci. Math.* (*Szeged*), **79** (2013), 273–287.

[14] E. Goya and T. Saitô, On intertwining by an operator having a dense range, *Tôhoku Math. J.*, **33** (1981), 127–131.

[15] P. A. Fillmore, *Notes on Operator Theory*, Van Nostrand, New York, 1970.

[16] P. R. Halmos, Quasitriangular operators, *Acta Sci. Math.* (*Szeged*), **29** (1968), 283–293.

[17] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887–933.

[18] P. R. Halmos, *A Hilbert Space Problem Book*, Springer, New York, 1982.

[19] P. R. Halmos and L. J. Wallen, Powers of partial isometries, *J. Math. Mech.*, **19** (1970), 657–663.

[20] D. A. Herrero, *Approximation of Hilbert Space Operators – Volume I*, Longman, Harlow, 1989.

[21] L. Kérchy, Isometric asymptotes of power bounded operators, *Indiana Univ. Math. J.*, **38** (1989), 173–188.

[22] L. Kérchy, Unitary asymptotes of Hilbert space operators, *Functional Analysis and Operator Theory, Warsaw*, 1992, Banach Center Publ. Vol. 30, Polish Acad. Sci., Warsaw, 1994, 191–201.

[23] L. Kérchy, Operators with regular norm-sequences, *Acta Sci. Math.* (*Szeged*), **63** (1997), 571–605.

[24] L. Kérchy, Generalized Toeplitz operators, *Acta Sci. Math.* (*Szeged*), **68** (2002), 373–400.

[25] L. Kérchy, Generalized Toeplitz operators associated with operators of regular norm-sequences, *Semigroup of Operators: Theory and Applications*, Optimization Software, Los Angeles, 2002, 119–131.

[26] M. Kosiek and L. Suciu, Decompositions and asymptotic limit for bicontraction, *Ann. Polon. Math.*, **105** (2012), 43–64.

[27] C. S. Kubrusly, Equivalent invariant subspace problems, *J. Operator Theory*, **38** (1997), 323–328.

[28] C. S. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Boston, 1997.

[29] C. S. Kubrusly, *Hilbert Space Operators: A Problem Solving Approach*, Birkhäuser, Boston, 2003.

[30] C. S. Kubrusly, *Spectral Theory of Operators on Hilbert Spaces*, Birkhäuser–Springer, New York, 2012.

[31] C. S. Kubrusly and B. P. Duggal, Contractions with $\mathcal{C}_{\cdot 0}$ direct summands, *Adv. Math. Sci. Appl.*, **11** (2001), 593–601.

[32] C. S. Kubrusly and N. Levan, Proper contractions and invariant subspaces, *Int. J. Math. Math. Sci.*, **28** (2001), 223–230.

[33] C. S. Kubrusly and P. C. M. Vieira, Strong stability for cohyponormal operators, *J. Operator Theory*, **31** (1994), 123–127.

[34] C. S. Kubrusly, P. C. M. Vieira and D. O. Pinto, A decomposition for a class of contractions, *Adv. Math. Sci. Appl.*, **6** (1996), 523–530.

[35] H. Langer, Ein Zerspaltungssatz für Operatoren im Hilbertraum, *Acta Math. Acad. Sci. Hungar.*, **12** (1961), 441–445.

[36] N. Levan, Canonical decompositions of completely nonunitary contractions, *J. Math. Anal. Appl.*, **101** (1984), 514–526.

[37] M. Martin and M. Putinar, *Lectures on Hyponormal Operators*, Birkhäuser, Basel, 1989.

[38] K. Okubo, The unitary part of paranormal operators, *Hokkaido Math. J.*, **6** (1977), 273–275.

[39] P. Pagacz, On Wold-type decomposition, *Linear Algebra Appl.*, **436** (2012), 3065–3071.

[40] C. M. Pearcy, *Some Recent Developments in Operator Theory*, CBMS Regional Conference Series in Mathematics No. 36, Amer. Math. Soc., Providence, 1978.

[41] V. Pták and P. Vrbová, An abstract model for compressions, *Časopis Pěst. Mat.*, **113** (1988), 252–266.

[42] C. R. Putnam, An inequality for the area of hyponormal spectra, *Math. Z.*, **116** (1970), 323–330.

[43] C. R. Putnam, Spectra of polar factors of hyponormal operators, *Trans. Amer. Math. Soc.*, **188** (1974), 419–428.

[44] C. R. Putnam, Hyponormal contractions and strong power convergence, *Pacific J. Math.*, **57** (1975), 531–538.

[45] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, Berlin, 1973; 2nd edn. Dover, New York, 2003.

[46] H. Radjavi and P. Rosenthal, The invariant subspace problem, *Math. Intelligencer*, **4** (1982), 33–37.

[47] P. Rosenthal, Equivalents of the invariant subspace problem, *Paul Halmos: Celebrating 50 Years of Mathematics*, Springer, New York, 1991, 179–188.

[48] A. L. Shields, Weighted shifts operators and analytic function theory, *Topics in Operator Theory* (Mathematical Surveys No. **13**, Amer. Math. Soc., Providence, 2nd pr. 1979), 49–128b.

[49] L. Suciu, Some invariant subspaces for A-contractions and applications, *Extracta Math.*, **21** (2006), 221–247.

[50] L. Suciu, Maximum A-isometric part of an A-contraction and applications, *Israel J. Math.*, **174** (2009), 419–443.

[51] L. Suciu and N. Suciu, Asymptotic behaviours and generalized Toeplitz operators, *J. Math. Anal. Appl.*, **349** (2009), 280–290.

[52] J. G. Stampfli and B. L. Wadhwa, An asymmetric Putnam-Fuglede theorem for dominant operators, *Indiana Univ. Math. J.*, **25** (1976), 359–365.

[53] B. Sz.-Nagy and C. Foiaş, Sur le contractions de l'espace de Hilbert IV, *Acta Sci. Math.* (*Szeged*), **21** (1960), 251–259.

[54] B. Sz.-Nagy and C. Foiaş, Sur le contractions de l'espace de Hilbert V. Translations bilatérales, *Acta Sci. Math.* (*Szeged*), **23** (1962), 106–129.

[55] B. Sz.-Nagy, C. Foiaş, H. Bercovici and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Springer, New York, 2010; (enlarged 2nd edn. of B. Sz.-Nagy and C. Foiaş, North-Holland, Amsterdam, 1970).

[56] T. Yoshino, On the unitary part of dominant contractions, *Proc. Japan Acad. Ser. A Math. Sci.*, **66** (1990), 272–273.

[57] T. Yoshino, *Introduction to Operator Theory*, Longman, Harlow, 1993.

[58] T. Yoshino, *The unitary part of $\mathcal{F}$ contractions*, Proc. Japan Acad. Ser. A Math. Sci., 75(1999), 50–52.

C. S. Kubrusly, Catholic University, 22453-900, Rio de Janeiro, RJ, Brazil; *e-mail*: carlos@ele.puc-rio.br