

Characterization of generalized quasi-arithmetic means

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Dedicated to the 80th birthday of Professor László Leindler

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Abstract. In this paper we characterize generalized quasi-arithmetic means, that is means of the form $M(x_1, \dots, x_n) := (f_1 + \dots + f_n)^{-1}(f_1(x_1) + \dots + f_n(x_n))$, where $f_1, \dots, f_n: I \rightarrow \mathbb{R}$ are strictly increasing and continuous functions. Our characterization involves the Gauss composition of the cyclic mean-type mapping induced by M and a generalized bisymmetry equation.

1. Introduction

The notion of quasi-arithmetic mean was introduced in the book of Hardy, Littlewood and Pólya in [12] as a function $A_f: \bigcup_{n=1}^{\infty} I^n \rightarrow I$ defined by

$$A_f(x_1, \dots, x_n) := f^{-1}\left(\frac{f(x_1) + \dots + f(x_n)}{n}\right) \quad (n \in \mathbb{N}, x_1, \dots, x_n \in I), \quad (1)$$

where $I \subseteq \mathbb{R}$ denotes a non-degenerated interval (also in the rest of this paper) and $f: I \rightarrow \mathbb{R}$ is a continuous strictly monotone function. The mean A_f is said to be the *quasi-arithmetic mean* generated by f . The restriction of A_f to I^n will be called the *n-variable quasi-arithmetic mean* generated by f .

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One can easily see that $M = A_f$ is a *mean* in the sense that, for all $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in I$,

$$\min(x_1, \dots, x_n) \leq M(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n)$$

holds. Furthermore, $M = A_f$ is a *strict mean* because both inequalities are strict whenever $\min(x_1, \dots, x_n) < \max(x_1, \dots, x_n)$.

For the equality problem of quasi-arithmetic means the following result can be established.

Theorem A. *Let $f, g: I \rightarrow \mathbb{R}$ be continuous and strictly monotone functions. Then the following properties are equivalent:*

- (i) *for all $n \in \mathbb{N}$ and for all $x_1, \dots, x_n \in I$,*

$$A_f(x_1, \dots, x_n) = A_g(x_1, \dots, x_n);$$

- (ii) *there exists $n \in \mathbb{N} \setminus \{1\}$ such that for all $x_1, \dots, x_n \in I$,*

$$A_f(x_1, \dots, x_n) = A_g(x_1, \dots, x_n);$$

- (iii) *the functions f and g are affine transformations of each other, that is, there exist real numbers $a \neq 0$ and b such that $g = af + b$.*

The characterization of quasi-arithmetic means was independently found by Kolmogorov [13], Nagumo [22] and de Finetti [11]. The result established by Kolmogorov reads as follows.

Theorem B. *A function $M: \bigcup_{n=1}^{\infty} I^n \rightarrow I$ is a quasi-arithmetic mean, that is, there exists a continuous strictly monotone function $f: I \rightarrow \mathbb{R}$ such that $M = A_f$ if and only if*

- (i) *for all $n \in \mathbb{N}$, the restriction $M_n := M|_{I^n}$ is a continuous and symmetric function on I^n which is strictly increasing in each of its variables;*
 (ii) *for all $n \in \mathbb{N}$, M_n is reflexive, that is, $M_n(x, \dots, x) = x$ for all $x \in I$;*
 (iii) *M is associative, that is, for all $n, m \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_m \in I$, we have*

$$M_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m) = M_{n+m}(x_1, \dots, x_n, y, \dots, y), \quad (2)$$

where $y = M_m(y_1, \dots, y_m)$.

The above characterization theorem does not characterize quasi-arithmetic means of a fixed number of variables because (2) involves m - and $(n+m)$ -variable means. The characterization of 2-variable quasi-arithmetic means was established by Aczél [2] and this was extended to the n -variable case by Münnich, Maksa and Mokken [20, 21]. Their results can be formulated in the following way.

Theorem C. *Let $n \geq 2$ and let $M: I^n \rightarrow I$. Then M is an n -variable quasi-arithmetic mean, that is, $M = A_f|_{I^n}$ for some continuous strictly monotone function $f: I \rightarrow \mathbb{R}$ if and only if*

- (i) *M is a continuous and symmetric function on I^n which is strictly increasing in each of its variables;*
- (ii) *M is reflexive;*
- (iii) *M is bisymmetric, that is, for all $x_{i,j} \in I$ ($i, j \in \{1, \dots, n\}$), we have*

$$\begin{aligned} M(M(x_{1,1}, \dots, x_{1,n}), \dots, M(x_{n,1}, \dots, x_{n,n})) \\ = M(M(x_{1,1}, \dots, x_{n,1}), \dots, M(x_{1,n}, \dots, x_{n,n})). \end{aligned} \quad (3)$$

It turns out that also weighted (and therefore, in general, non-symmetric) quasi-arithmetic means can be characterized by the bisymmetry equation (3).

Quasi-arithmetic means can be generalized in several ways. In 1963 Bajraktarević [7] introduced the notion of quasi-arithmetic means weighted by a weight function. Their equality problem was solved by Aczél and Daróczy in [3]. The characterization theorem of Bajraktarević means was found by Páles in [24]. Another (symmetric) generalization, the notion of deviation mean, was invented by Daróczy in [8, 9]. The characterization of the Daróczy means was then established by Páles in [23]. Both of these characterization theorems use systems of functional inequalities instead of functional equations like associativity or bisymmetry.

In this paper we consider a recent generalization of quasi-arithmetic means which was introduced by Matkowski [19] in 2010. Given a system $f_1, \dots, f_n: I \rightarrow \mathbb{R}$ of continuous strictly increasing functions, the *generalized n -variable quasi-arithmetic mean* $A_{f_1, \dots, f_n}: I^n \rightarrow I$ is defined by

$$A_{f_1, \dots, f_n}(x_1, \dots, x_n) := (f_1 + \dots + f_n)^{-1}(f_1(x_1) + \dots + f_n(x_n)) \quad (x_1, \dots, x_n \in I).$$

The functions $f_1, \dots, f_n$ are called the generators of the mean $A_{f_1, \dots, f_n}$. In the particular case $f_1 = \dots = f_n = f$, one can see that $A_{f_1, \dots, f_n}$ reduces to the quasi-arithmetic mean A_f . More generally, if $f_i = \lambda_i f$, where $\lambda_1, \dots, \lambda_n > 0$ and $f: I \rightarrow \mathbb{R}$ is a continuous strictly increasing function, then $A_{f_1, \dots, f_n}$ will be equal to a so-called weighted quasi-arithmetic mean. One can easily check that generalized n -variable quasi-arithmetic means are strict means.

The equality problem of generalized n -variable quasi-arithmetic mean was answered by Matkowski in [19] as follows.

Theorem D. *Let $f_1, \dots, f_n, g_1, \dots, g_n: I \rightarrow \mathbb{R}$ be continuous strictly increasing functions. Then the following two assertions are equivalent:*

(i) for all $x_1, \dots, x_n \in I$,

$$A_{f_1, \dots, f_n}(x_1, \dots, x_n) = A_{g_1, \dots, g_n}(x_1, \dots, x_n);$$

(ii) there exist real numbers $a \neq 0$ and $b_1, \dots, b_n$ such that, for all $i \in \{1, \dots, n\}$,

$$g_i = af_i + b_i.$$

The main problem addressed in this paper is the characterization of generalized n -variable quasi-arithmetic means. For this purpose, we recall the notion of Gauss composition in the next section with its basic properties and, in the last section, we introduce the family of cyclic mean-type mapping attached to a given generalized n -variable quasi-arithmetic mean and we compute its Gauss composition explicitly. Using this, we shall deduce a bisymmetry type identity for generalized n -variable quasi-arithmetic means which will turn out to be the key characteristic property beyond regularity and reflexivity properties. The key point in our proof is the use of the description of the CM -solutions of the generalized bisymmetry equation due to Maksa [14].

2. Auxiliary notions and results

2.1. Gauss iteration and Gauss composition of means

Given a system $M = (M_1, \dots, M_n): I^n \rightarrow I^n$ of n -variable means (which is also called an n -variable mean-type mapping (cf. [18]) and an element $x \in I^n$, the sequence $x_k := M^k(x)$ is called the *Gauss iteration* of $x \in I^n$ by the mean-type mapping M .

Theorem E. Assume that $M = (M_1, \dots, M_n): I^n \rightarrow I^n$ is a continuous strict n -variable mean-type mapping, that is, $M_1, \dots, M_n$ are continuous and strict means. Then there exists a unique continuous strict mean $K: I^n \rightarrow I$ such that the sequence of iterates (M^k) converges pointwise to the mean-type mapping $(K, \dots, K)$. Furthermore, K is the unique mean satisfying the identity

$$K \circ M = K, \tag{4}$$

which is called the *invariance equation* for K .

For the proof of this result, the reader should check the papers [10], [18].

The mean K constructed in the above theorem will be called the *Gauss composition of the means* $(M_1, \dots, M_n)$ and will be denoted by $\Gamma(M_1, \dots, M_n)$.

2.2. Cyclic mean-type mappings

Given $n \geq 2$, define the cyclic permutation $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ by

$$\sigma(k) := \begin{cases} n, & \text{if } k = 1, \\ k - 1, & \text{if } k \in \{2, \dots, n\}. \end{cases}$$

Clearly, $\sigma^n = \sigma^0$ which is the identity mapping.

In the proof of the main result we shall need the following

Lemma 1. *For all $i \in \{1, \dots, n\}$, we have $\sigma^i(i) = n$.*

Proof. The statement is obvious for $i = 1$ by the definition of σ . Assume that it holds for $i = k$, where $k \in \{1, \dots, n-1\}$. Then using this inductive assumption, we get

$$\sigma^{k+1}(k+1) = \sigma^k(\sigma(k+1)) = \sigma^k(k) = n.$$

Thus, the statement is also true for $i = k+1$. ■

For an n -variable mean $M: I^n \rightarrow I$ and an index $i \in \mathbb{Z}$, the i th cyclically permuted mean $M^{(i)}: I^n \rightarrow I$ is defined by

$$M^{(i)}(x_1, \dots, x_n) = M(x_{\sigma^i(1)}, \dots, x_{\sigma^i(n)}) \quad (x_1, \dots, x_n \in I).$$

The mapping $(M^{(0)}, \dots, M^{(n-1)}): I^n \rightarrow I^n$ is said to be the *cyclic mean-type mapping induced by M* .

Proposition 2. *If $M: I^n \rightarrow I$ is a continuous strict n -variable mean, then the Gauss composition $\Gamma(M^{(0)}, \dots, M^{(n-1)})$ is a cyclically symmetric mean, i.e., for all $i \in \mathbb{Z}$,*

$$\Gamma(M^{(0)}, \dots, M^{(n-1)}) = (\Gamma(M^{(0)}, \dots, M^{(n-1)}))^{(i)}.$$

Proof. Put $K = \Gamma(M^{(0)}, \dots, M^{(n-1)})$. By Theorem E, K is the unique n variable mean which solves the functional equation

$$K \circ (M^{(0)}, \dots, M^{(n-1)}) = K. \tag{5}$$

Therefore, for $i \in \mathbb{Z}$,

$$\begin{aligned} K^{(i)} \circ (M^{(0)}, \dots, M^{(n-1)}) &= K \circ (M^{(i)}, \dots, M^{(i+n-1)}) \\ &= (K \circ (M^{(0)}, \dots, M^{(n-1)}))^{(i)} = K^{(i)}. \end{aligned}$$

Thus, $K^{(i)}$ is also a solution of the invariance equation (5). Hence, by the unique solvability, it follows that $K = K^{(i)}$. ■

2.3. *CM*-quasi-sums, *CM*-functions and generalized bisymmetry

An n -variable function $F: I^n \rightarrow I$ is called a *CM-quasi-sum* (cf. Maksa [14]) if there exist *CM* (i.e., continuous and strictly increasing) functions $f_1, \dots, f_n: I \rightarrow \mathbb{R}$ and $f: f_1(I) + \dots + f_n(I) \rightarrow I$ such that

$$F(x_1, \dots, x_n) = f(f_1(x_1) + \dots + f_n(x_n)) \quad ((x_1, \dots, x_n) \in I^n).$$

The functions $f, f_1, \dots, f_n$ are called the generators of the quasi-sums. A function $F: I^n \rightarrow I$ is said to be a *CM*-function if it is continuous and strictly increasing in each of its variables.

The following result, which is a particular case of the general theorem of Maksa [14], will play a crucial role in our approach.

Theorem F. *Let $n, m > 1$. Let $F, F_1, \dots, F_n: I^m \rightarrow I$ and $G, G_1, \dots, G_m: I^n \rightarrow I$ be *CM*-functions such that, for all $x_{i,j} \in I$, ($i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$),*

$$\begin{aligned} F(G_1(x_{1,1}, \dots, x_{n,1}), \dots, G_m(x_{1,m}, \dots, x_{n,m})) \\ = G(F_1(x_{1,1}, \dots, x_{1,m}), \dots, F_n(x_{n,1}, \dots, x_{n,m})). \end{aligned}$$

*Then $F, F_1, \dots, F_n$ and $G, G_1, \dots, G_m$ are *CM*-quasi-sums.*

Lemma 3. *A *CM*-quasi-sum $M: I^n \rightarrow I$ is reflexive if and only if it is a generalized n -variable quasi-arithmetic mean.*

Proof. Obviously generalized n -variable quasi-arithmetic mean are *CM*-quasi-sums.

Assume now that a *CM*-quasi-sum

$$M(x_1, \dots, x_n) = f(f_1(x_1) + \dots + f_n(x_n)) \quad ((x_1, \dots, x_n) \in I^n)$$

is a mean. Setting $x_1 = \dots = x_n = x$, by the reflexivity of M , we get $f((f_1 + \dots + f_n)(x)) = x$ for all $x \in I$, whence

$$f = (f_1 + \dots + f_n)^{-1}.$$

Thus, M is of the form $A_{f_1, \dots, f_n}$, i.e., M is a generalized n -variable quasi-arithmetic mean. ■

3. Characterization of generalized quasi-arithmetic means

Theorem 4. Let $f_1, \dots, f_n: I \rightarrow \mathbb{R}$ be continuous strictly increasing functions. Then the Gauss composition of the means $A_{f_1, \dots, f_n}^{(0)}, \dots, A_{f_1, \dots, f_n}^{(n-1)}$ is the quasi-arithmetic mean $A_{f_1 + \dots + f_n}$. Furthermore, for all $x_{i,j} \in I$, $(i, j \in \{1, \dots, n\})$, the generalized bisymmetry equation

$$\begin{aligned} & A_{f_1 + \dots + f_n} (A_{f_1, \dots, f_n}^{(0)}(x_{1,1}, \dots, x_{n,1}), \dots, A_{f_1, \dots, f_n}^{(n-1)}(x_{1,n}, \dots, x_{n,n})) \\ &= A_{f_1 + \dots + f_n} (A_{f_1, \dots, f_n}^{(0)}(x_{1,1}, \dots, x_{1,n}), \dots, A_{f_1, \dots, f_n}^{(n-1)}(x_{n,1}, \dots, x_{n,n})) \end{aligned} \quad (6)$$

holds.

Proof. First we prove that (6) is satisfied.

Let $x_{i,j} \in I$ for $i, j \in \{1, \dots, n\}$ and define, for $i \in \{0, \dots, n-1\}$,

$$y_i := A_{f_1, \dots, f_n}^{(i)}(x_{1,i+1}, \dots, x_{n,i+1}) \quad \text{and} \quad z_i := A_{f_1, \dots, f_n}^{(i)}(x_{i+1,1}, \dots, x_{i+1,n}).$$

Then

$$\begin{aligned} y_i &= A_{f_1, \dots, f_n}(x_{\sigma^i(1), i+1}, \dots, x_{\sigma^i(n), i+1}) = (f_1 + \dots + f_n)^{-1} \left(\sum_{j=1}^n f_j(x_{\sigma^i(j), i+1}) \right), \\ z_i &= A_{f_1, \dots, f_n}(x_{i+1, \sigma^i(1)}, \dots, x_{i+1, \sigma^i(n)}) = (f_1 + \dots + f_n)^{-1} \left(\sum_{j=1}^n f_j(x_{i+1, \sigma^i(j)}) \right). \end{aligned}$$

Therefore, for the left- and right-hand sides of (6), we obtain the following expressions:

$$\begin{aligned} A_{f_1 + \dots + f_n}(y_0, \dots, y_{n-1}) &= (f_1 + \dots + f_n)^{-1} \left(\frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=1}^n f_j(x_{\sigma^i(j), i+1}) \right) \\ &= (f_1 + \dots + f_n)^{-1} \left(\frac{1}{n} \sum_{\alpha=1}^n \sum_{\beta=1}^n f_{\sigma^{1-\beta}(\alpha)}(x_{\alpha, \beta}) \right), \\ A_{f_1 + \dots + f_n}(z_0, \dots, z_{n-1}) &= (f_1 + \dots + f_n)^{-1} \left(\frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=1}^n f_j(x_{i+1, \sigma^i(j)}) \right) \\ &= (f_1 + \dots + f_n)^{-1} \left(\frac{1}{n} \sum_{\alpha=1}^n \sum_{\beta=1}^n f_{\sigma^{1-\alpha}(\beta)}(x_{\alpha, \beta}) \right). \end{aligned}$$

Thus, in order that (6) be satisfied, it is sufficient to show that

$$\sigma^{1-\beta}(\alpha) = \sigma^{1-\alpha}(\beta). \quad (7)$$

However, by Lemma 1, we have that $\sigma^\alpha(\alpha) = n = \sigma^\beta(\beta)$ holds for all $\alpha, \beta \in \{1, \dots, n\}$. Applying the map $\sigma^{1-\alpha-\beta}$ to both sides of this equation, we get

$$\sigma^{1-\beta}(\alpha) = \sigma^{1-\alpha-\beta}(n) = \sigma^{1-\alpha}(\beta),$$

which proves (7) and hence identity (6) is also verified.

To prove that the Gauss composition of the means $A_{f_1, \dots, f_n}^{(0)}, \dots, A_{f_1, \dots, f_n}^{(n-1)}$ is the quasi-arithmetic mean $A_{f_1 + \dots + f_n}$, substitute $x_{i,j} := y_i$ into (6) where $y_1, \dots, y_n \in I$. Then (6) simplifies to

$$A_{f_1 + \dots + f_n}(A_{f_1, \dots, f_n}^{(0)}(y_1, \dots, y_n), \dots, A_{f_1, \dots, f_n}^{(n-1)}(y_1, \dots, y_n)) = A_{f_1 + \dots + f_n}(y_1, \dots, y_n).$$

Therefore $K = A_{f_1 + \dots + f_n}$ is the solution of the invariance equation

$$K \circ (A_{f_1, \dots, f_n}^{(0)}, \dots, A_{f_1, \dots, f_n}^{(n-1)}) = K,$$

hence, by the unique solvability of invariance equations, K is the Gauss composition of the means $A_{f_1, \dots, f_n}^{(0)}, \dots, A_{f_1, \dots, f_n}^{(n-1)}$. ■

The following result is our main characterization theorem.

Theorem 5. *A function $M: I^n \rightarrow I$ is a generalized quasi-arithmetic mean if and only if*

- (i) *M is a CM-function on I^n ;*
- (ii) *M is reflexive;*
- (iii) *$\Gamma(M^{(0)}, \dots, M^{(n-1)})$ is a CM-function and M is generalized bisymmetric, that is, for all $x_{i,j} \in I$ ($i, j \in \{1, \dots, n\}$), we have*

$$\begin{aligned} & \Gamma(M^{(0)}, \dots, M^{(n-1)})(M^{(0)}(x_{1,1}, \dots, x_{1,n}), \dots, M^{(n-1)}(x_{n,1}, \dots, x_{n,n})) \\ &= \Gamma(M^{(0)}, \dots, M^{(n-1)})(M^{(0)}(x_{1,1}, \dots, x_{n,1}), \dots, M^{(n-1)}(x_{1,n}, \dots, x_{n,n})). \end{aligned} \quad (8)$$

Proof. If M is a generalized quasi-arithmetic mean of the form $A_{f_1, \dots, f_n}$, then M is a reflexive CM-function and, by Theorem 4, the Gauss composition of the means $M^{(0)}, \dots, M^{(n-1)}$ is the quasi-arithmetic mean $A_{f_1 + \dots + f_n}$ furthermore (6) is satisfied, which is now equivalent to (8).

Now assume that M is a reflexive CM-function which satisfies (8). Then, using Theorem F, it follows that M is a CM-quasi-sum. Due to its reflexivity, by Lemma 3, we obtain that M is a generalized quasi-arithmetic mean. ■

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